Adapting Trigonometric Substitution — Yigal Kamel

The goal for this method is to make a good substitution to compute integrals which contain one of the following expressions:

Ⓐ $\sqrt{k^2 - x^2}$ Ⓑ $\sqrt{x^2 - k^2}$ Ⓒ $\sqrt{k^2 + x^2}$ (k = constant)

Strategy: Use the pythagorean theorem as our guide.

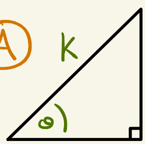
$$a^2 + b^2 = c^2 \rightsquigarrow c = \sqrt{a^2+b^2}$$

or: $$a^2 = c^2 - b^2 \rightsquigarrow a = \sqrt{c^2 - b^2}.$$

These equations replace expressions like Ⓐ, Ⓑ, & Ⓒ with something simpler.

Now, we just have to label a right triangle in a good way.

Ⓐ

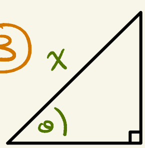


$\sqrt{k^2-x^2} = k\sin\theta$

$x = k\cos\theta \Rightarrow dx = -k\sin\theta\, d\theta$

$$\boxed{\begin{aligned} x &= k\cos\theta \\ dx &= -k\sin\theta\, d\theta \\ \sqrt{k^2-x^2} &= k\sin\theta \end{aligned}}$$

Ⓑ



$\sqrt{x^2-k^2} = x\sin\theta$

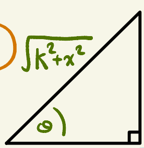
$= k\sec\theta\sin\theta = k\tan\theta$

$k = x\cos\theta \Rightarrow x = \dfrac{k}{\cos\theta} = k\sec\theta$

$\Rightarrow dx = k\sec\theta\tan\theta\, d\theta$

$$\boxed{\begin{aligned} x &= k\sec\theta \\ dx &= k\sec\theta\tan\theta\, d\theta \\ \sqrt{x^2-k^2} &= k\tan\theta \end{aligned}}$$

Ⓒ



$x = \sqrt{k^2+x^2}\,\sin\theta = \dfrac{k}{\cos\theta}\sin\theta = k\tan\theta$

$k = \sqrt{k^2+x^2}\,\cos\theta \Rightarrow \sqrt{k^2+x^2} = \dfrac{k}{\cos\theta} = k\sec\theta$

$x = k\tan\theta \Rightarrow dx = k\sec^2\theta\, d\theta$

$$\boxed{\begin{aligned} x &= k\tan\theta \\ dx &= k\sec^2\theta\, d\theta \\ \sqrt{k^2+x^2} &= k\sec\theta \end{aligned}}$$

NOTE: In any of the calculations with triangles to the left, we could switch the roles of the legs of the triangle & get a different substitution that can work (this is the same as calling the other non-right angle "θ").