THE PHONY MULTIPLICATION ON QUILLEN'S K-THEORY

REFERENCES

- Thomason, "Beware the phony multiplication on Quillen's $\mathcal{A}^{-1}\mathcal{A}$".
- Quillen[-Grayson], "Higher algebraic K-theory I & II".
- Gillet-Grayson, "The loop space of the Q-construction".
- Segal, "K-homology theory and algebraic K-theory".
- Hohnhold-Stolz-Teichner, "From minimal geodesics to supersymmetric field theories."

The integers

$(\mathbb{N}, +)$ = commutative monoid

Want "subtraction" $\rightsquigarrow$ abelian group

$$\mathbb{Z} = \{a-b \mid a, b \in \mathbb{N}\}/\sim$$

generated by $a-b \sim c-d$ if $\exists n$ w/ $\begin{cases} a+n = c \\ b+n = d \end{cases}$

$$= F(\mathbb{N}) \big/ a +_F b \sim a +_{\mathbb{N}} b$$

Notice: $(\mathbb{N}, +, \cdot)$ = com. semiring (or rig)

& the operation $\cdot$ automatically extends to a ring structure on $\mathbb{Z}$:

$$(a-b)\cdot(c-d) = (a\cdot c + b\cdot d) - (a\cdot d + b\cdot c)$$

Grothendieck's group completion

The above works for any commutative (semiring) monoid.

$(M, \oplus)$ = com. monoid.

$\Rightarrow K(M) = F(M) / a+b \sim a \oplus b$

$= \{ a-b \mid a, b \in M \} / \sim$

generated by $a-b \sim c-d$ if $\exists n$ w/ $\begin{cases} a+n = c \\ b+n = d \end{cases}$

if $(M, \oplus, \otimes)$ is a semiring, then $K(M)$ becomes a ring.

Classical algebraic K-theory

Study a ring via its category of modules.

$(\pi_0 \mathrm{Mod}_R^{\simeq}, \oplus, \otimes)$ = semiring

f.g. projective

$\Rightarrow$ ring $K_0(R) = K(\pi_0 \mathrm{Mod}_R^{\simeq})$.

Ex: R = field, $K_0(R) \cong \mathbb{Z}$.

Ethos of stable homotopy theory: Given

$$\begin{tikzcd}
\left(\begin{array}{c}\text{highly structured}\\ \text{object}\end{array}\right) \arrow[rr, mapsto] \arrow[dr, dashed, "\text{Try}"'] & & A \in \mathrm{Ab} \\
& X \in \mathrm{Sp} \arrow[ur, mapsto, "\pi_0"'] &
\end{tikzcd}$$

$\Rightarrow$ get more (hopefully computable) invariants $\pi_* X$.

Want:
- $R \longmapsto K(R) \in \mathrm{Sp}$ with $\pi_0 K(R) \cong K_0(R)$

  $\Rightarrow K_i R$ (Quillen, Segal, Waldhausen, + ...)
- $\otimes$ on $\mathrm{Mod}_R \Longrightarrow$ ring spectrum

Idea: group complete the category Mod_R

Quillen's $S^{-1}S$-construction

Generalizes Grothendieck's group completion

$$\begin{pmatrix}\text{commutative} \\ \text{monoids}\end{pmatrix} \rightsquigarrow \begin{pmatrix}\text{sym. monoidal} \\ \text{categories}\end{pmatrix}$$

e.g. Mod_R.

<u>Def</u>: Given a symmetric monoidal cat. $(S, \oplus)$, define a sym. mon. cat $S^{-1}S$ with

<u>obj</u>: (A, B) $\quad A, B \in \operatorname{obj} S$.

<u>mor</u>: $(N, \alpha, \beta) : (A, B) \longrightarrow (C, D)$ where

- $N \in \operatorname{obj} S$
- $\alpha : A \oplus N \longrightarrow C$
- $\beta : B \oplus N \longrightarrow D$

modulo an equiv. of (N, α, β)'s composition : $(A,B) \xrightarrow{(N,\alpha,\beta)} (C,D) \xrightarrow{(M,\gamma,\delta)} (E,F)$

$$(N \oplus M,\ \gamma \circ (\alpha \oplus id_M),\ \delta \circ (\beta \oplus id_M))$$

i.e.

$$\begin{tikzcd} A \oplus N \oplus M \arrow[rr] \arrow[dr, "\alpha \oplus id_M"'] & & E \\ & C \oplus M \arrow[ur, "\gamma"'] & \end{tikzcd} \qquad \begin{tikzcd} B \oplus N \oplus M \arrow[rr] \arrow[dr, "\beta \oplus id_M"'] & & F \\ & D \oplus M \arrow[ur, "\delta"'] & \end{tikzcd}$$

monoidal structure : $(A,B) \oplus (C,D) = (A \oplus C,\ B \oplus D)$

functorial : e.g. $f : C \to C'$

$$\Rightarrow \quad (A \oplus C,\ B \oplus D) \xrightarrow[(0,\ id_A \oplus f,\ id_{B \oplus D})]{} (A \oplus C',\ B \oplus D)$$

- $\iota : S \longrightarrow S^{-1}S$

  $A \longmapsto (0, A)$

Thm: (Quillen) Suppose S is a sym. monoidal groupoid & $\forall A \in S$, $A \oplus - : S \to S$ is faithful.
Then
$$B\iota : BS \longrightarrow BS^{-1}S$$
is a (topological) group completion of H-spaces, i.e.

- $\pi_0 B\iota : \pi_0 BS \longrightarrow \pi_0 BS^{-1}S$ is group completion of the monoid $(\pi_0 BS, \oplus)$,
- $H_*(BS^{-1}S)$ is the localization of $H_*(BS)$ w.r.t. action of $\pi_0 BS$.

$\Rightarrow$ $K(R) := B(\mathrm{Mod}_R^{\simeq})^{-1}(\mathrm{Mod}_R^{\simeq})$ as a connective spectrum.

Q: $\otimes$ on Mod_R $\Rightarrow$ ring spectrum $K(R)$?

A: (Thomason) yes & no. We'll come back to this.

Topological K-theory

Like above, but replace "modules over a ring" with "modules over a space" = vector bundles.

$$K(X) = K(\pi_0 \operatorname{Vect}_{\mathbb{R} \text{ or } \mathbb{C}}(X)^{\simeq}) \cong K_0(\operatorname{Vect}_{\mathbb{R} \text{ or } \mathbb{C}}(X)^{\simeq})$$

<u>Rmk</u>: Swan's Thm $\Rightarrow$ $K(X) = K_0(C(X))$ (under suff. hypotheses)

so (w/ fixed X) top'l K-th. an example of alg'c K-th.

<u>What's different?</u> Let X vary...

Bott Periodicity $\Rightarrow$ $X \longmapsto K(X)$

extends to a cohomology theory K^*

$\Rightarrow$ represented by a (ring) spectrum $= \begin{cases} KO, & \mathbb{R} \\ KU, & \mathbb{C}. \end{cases}$

(independent of X!)

Q: Are KU &/or KO examples of $K(R)$ for some ring R?

A: Not really, they're not connective.

Non-connective spectra can be thought of as a sequence of connective spectra (∞-loop spaces) related in a specified way.

Q: Can KU &/or KO be thought of as a sequence of $K(R_i)$'s related in a specified way?

A: Kind of! Maybe yes... though I am still searching for the best formulation.

The Atiyah-Bott-Shapiro isomorphism

- $Cl_n = T(\mathbb{R}^n)/(v^2 + \|v\|^2)$ "Clifford algebra"
- $Cl_n \otimes Cl_m \simeq Cl_{n+m}$ (in super algebras)
- $N \in Mod_{Cl_n}$, $M \in Mod_{Cl_m}$

  $\Rightarrow N \otimes M \in Mod_{Cl_{n+m}}$.
- $M(Cl_*) := K(\pi_0 Mod_{Cl_*}^{\simeq})$ is a graded ring.
- $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1} \Rightarrow Cl_n \xhookrightarrow{i} Cl_{n+1} \Rightarrow M(Cl_{n+1}) \xrightarrow{i^*} M(Cl_n)$
- $A_n := M(Cl_n)/_{i^*}M(Cl_{n+1})$

  $\Rightarrow (A_*, \oplus, \otimes)$ is a graded ring

$\mathbb{C}l_n := Cl_n \otimes \mathbb{C}$

$\rightsquigarrow A^c_*$

$\Rightarrow A^c_* \cong KU_*$

[**Thm:** (ABS) $A_* \cong \pi_* KO$ iso of rings.

Q: Can we enhance this construction to realize ① KO, ② with its ring structure, from the perspective of algebraic K-theory of Clifford modules?

① **A1:** (Rezk, MO) $KO \simeq D_1^{\text{weiss}}\left(V \mapsto K(\mathrm{Mod}_{C\ell(V)})\right)$

A2: (Hebestreit–Stolz–Teichner) Define (top'l) category D_n in the spirit of Quillen's $S^{-1}S$.

obj: finite dim'l ($\mathbb{Z}/2$-graded) $C\ell_n$-modules

mor: $W_0 \xrightarrow{(i,e)} W_1$, $i: W_0 \hookrightarrow W_1$

(sign choices abound!)

& e = odd self-adj. operator on $W_1 \ominus W_0$, $e^2 = \pm 1$

$\Longleftrightarrow$ extension of $W_1 \ominus W_0$ to a $C\ell_{n+1}$-module.

Evidently, $\pi_0 BD_n = A_n$

[<u>Thm</u>: (HST) $BD_n \simeq KO_n$.

<u>Q</u>: Spectrum structure? In progress...

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② <u>A1</u>: Recent preprint 9/26/23 "Monoidal structures in orthogonal calculus"? need to investigate.

<u>A2</u>: Want functors

$$D_p \times D_q \longrightarrow D_{p+q}$$

$$M, N \longmapsto M \otimes N$$

But this proves difficult on morphisms...

$N_2 \oplus N_1$	f_2	?	$e_2 \otimes f_2$
$N_1 \oplus N_0$	f_1	$e_1 \otimes f_1$	?
N_0		e_1	e_2
	M_0	$M_1 \oplus M_0$	$M_2 \oplus M_1$

There are a few reasonable choices for what goes here, but none are functorial!

Thomason explains that there exists no such functor since the twist map

$$M \otimes M \longrightarrow M \otimes M$$

is not the identity.

Back to alg. K-th.

[Thomason's claim: The $\otimes$ on $\mathrm{Mod}_R = S$ does not descend to $S^{-1}S$.

② What I did above, no def. is functorial.

① [Fiedorowicz '78] Tries to construct product on $K(R)$ via a universal property which is deduced from the following (false)

[Claim: Let $T : S^{-1}S \to S^{-1}S$ be the twist functor $T : (A,B) \mapsto (B,A)$. Then $\exists$ nat. trans.

$$\eta : 0 \longrightarrow T \oplus \mathrm{id}$$

need $\eta(A,B) : (0,0) \longrightarrow (B \oplus A, A \oplus B)$

obvious candidate : $\overset{\shortparallel}{(A \oplus B, \tau, \mathrm{id}_{A \oplus B})}$

[**Prop**: η as defined above is not natural.

sketch: diagram chase to see

natural $\Leftrightarrow$ $(\tau : A \otimes A \to A \otimes A) = id_A$ //

One way to fix Thomason's observation...

[Gillet-Grayson]

- Generalize Quillen's $S^{-1}S$ to "categories with exact sequences", but now the result is a simplicial set GS instead of a category (but with the same objects as $S^{-1}S$).
- Construct a new operation $\mathcal{G}$ which increases the number of simplicial directions but preserves

homotopy type.

- Show $\mathcal{G}G \simeq G\mathcal{G} \Rightarrow GG$
- $GS \times GS \longrightarrow GGS$

$$(A,B) \times (C,D) \longmapsto \begin{pmatrix} A\otimes C & A\otimes D \\ B\otimes C & B\otimes D \end{pmatrix}$$

$$\Rightarrow K(R) \wedge K(R) \longrightarrow K(R).$$