Higher Semiadditivity (Talk notes, 1/24/24)

$\mathcal{C}$ = Category w/ initial $\emptyset$ & terminal $*$.

$$\begin{tikzcd}
\Rightarrow \exists!\ \mathrm{Nm}_{-1} : \emptyset \arrow[r, "\overset{?}{\sim}"] \arrow[d, equal] & * \arrow[d, equal] \\
\operatorname{colim}_{\emptyset} \arrow[r] & \lim_{\emptyset}
\end{tikzcd}$$

$\emptyset$ = (-1)-truncated space

pointed: yes	no
Group	Set
Ab	$\mathcal{S}$
Vect_k	
$\mathcal{S}_*$	
Sp	
all stable ∞-cat's	

Consequences of pointedness

- $\forall\ X, Y \in \mathcal{C}$ i.e. $\emptyset \to \mathrm{Map}_{\mathcal{C}}(X,Y)$ $\exists$ zero map

$$\int_{\emptyset} ! = 0 : X \xrightarrow{!} * \xrightarrow{\mathrm{Nm}_{-1}^{-1}} \emptyset \xrightarrow{!} Y$$

$\Longrightarrow$ enriched in pointed sets.

- If $\mathcal{C}$ has finite (co)products: $\forall\ X, Y \in \mathcal{C}$, $\exists$

$$\begin{tikzcd}
\mathrm{Nm}_0 : X \amalg Y \arrow[r, "\overset{?}{\sim}", "\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{pmatrix}"'] \arrow[d, equal] & X \times Y \arrow[d, equal] \\
\operatorname{colim}_{\{x,y\}} \arrow[r] & \lim_{\{x,y\}}
\end{tikzcd}$$

$\{x,y\}$ = (finite) 0-truncated space

semiadditive: yes	no
Ab	Set
Vect_k	$\mathcal{S}$
Sp	Group
all stable ∞-cat's	$\mathcal{S}_*$

Consequences of semiadditivity (repeat above)

- (use Nm_0^{-1}) $\forall\, f, g : X \to Y \quad \exists\, f+g$

$$\begin{tikzcd} X \arrow[r, "f\times g"] \arrow[rrr, bend right, "f+g"'] & Y\times Y \arrow[r, "\mathrm{Nm}_0^{-1}"] & Y \amalg Y \arrow[r, "\mathrm{id}\times\mathrm{id}"] & Y \end{tikzcd}$$

Further, since $\coprod_{i=1}^{n} X_i \xrightarrow{\sim} \prod_{i=1}^{n} X_i$

$$\Rightarrow \left(\underbrace{f_1, \dots, f_n : X \to Y}_{\{1,\dots,n\} \to \mathrm{Map}_C(X,Y)} \quad \Rightarrow \quad \exists \underbrace{\sum_{i=1}^{n} f_i}_{= \int_{\{1,\dots,n\}} f} : X \to Y \right)$$

$\Longrightarrow$ enriched in $\mathrm{CAlg}(S)$

$\Longrightarrow$ multiplication by n:

$$\underbrace{\mathrm{id}_X, \dots, \mathrm{id}_X}_{n \,=\, |\{1,\dots,n\}|} : X \to X \quad \Longrightarrow \quad \underbrace{n\cdot}_{|\{1,\dots,n\}|} = \sum_{i=1}^{n} \mathrm{id}_X : X \to X$$

Def: $m \geq -1$. A space A is <u>m-finite</u> if

$$\pi_k A = \begin{cases} \text{finite}, & k \leq m \\ 0, & k > m. \end{cases}$$

A space A is <u>π-finite</u> if A is m-finite for some m.

- 1-finite space $\to$ BG, G = finite group.

$$\left(\begin{matrix}\text{functor} \\ BG \xrightarrow[F]{} \mathcal{C}\end{matrix}\right) \iff \left(G \circlearrowright X \in \mathcal{C} \right) \quad \text{w/}$$

$$\operatorname*{colim}_{BG} F \simeq X_{hG} \quad \& \quad \lim_{BG} F \simeq X^{hG}$$

<u>Want</u>: $\quad Nm_1 : X_{hG} \longrightarrow X^{hG}$

Roughly,

$$\begin{tikzcd} X \arrow[r, "\sum_{g \in G} g"] \arrow[d] & X \\ X_{hG} \arrow[r, dashed, "\text{hope}"] & X^{hG} \arrow[u] \end{tikzcd}$$

<u>Ex</u>: (1-cat'l) $\mathcal{C} = Ab \ni A \circlearrowleft G$

$\Rightarrow \sum_{g \in G} g$ well-defined on orbits & image $\subseteq A^G$

$$\Rightarrow \quad \begin{tikzcd} Nm_1 : A_G \arrow[r] & A^G \arrow[l, bend right, dashed, "|G|^{-1}"'] \end{tikzcd} \quad \rightsquigarrow \quad \text{Not usually iso!}$$

e.g. $C_2 \circlearrowright \mathbb{Z}/2$ trivial $\Rightarrow Nm_1(x) = \sum_{g \in C_2} x = 2x = 0$.

if $\mathcal{C} = Vect_{\mathbb{Q}}$, then $Nm_1 : V_G \xrightarrow{\sim} V^G$.

<u>Def</u>: A (0-)semiadditive category $\mathcal{C}$ is <u>1-semiadditive</u> if $Nm_1 : X_{hG} \xrightarrow{\sim} X^{hG}$ iso $\forall$ finite $G \circlearrowright X \in \mathcal{C}$ (any 1-finite space is $\simeq \coprod_{i=1}^{n} BG_i$)

1-semiadditive

yes	no
$\mathrm{Vect}_{\mathbb{Q}}$, $\mathrm{Sp}_{H\mathbb{Q}}$	most others.

Within Sp: 1-semiadditivity is closely related with chromatic phenomena.

Thm:
(1) [Greenlees-Hovey-Sadofsky] $\mathrm{Sp}_{K(n)}$ is 1-semiadditive.
(2) [Kuhn] $\mathrm{Sp}_{T(n)}$ is 1-semiadditive.

$\hookrightarrow$ $Nm_1 : X_{hG} \longrightarrow X^{hG}$ iso

$\Leftrightarrow$ $\mathrm{cof}(N_1) \simeq X^{tG} \simeq 0$.

Thm: [Carmeli-Schlank-Yanovski] $0 \neq R$ p-local homotopy ring spectrum.

$$\left(\mathrm{Sp}_R \text{ is } 1\text{-semiadditive}\right) \Longleftrightarrow \left(\begin{array}{c}\exists\, n \geq 0 \text{ s.t.} \\ \mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}_R \subseteq \mathrm{Sp}_{T(n)}\end{array}\right)$$

Consequences of 1-semiadditivity

$$BG \xrightarrow{f} \mathrm{Map}_{\mathcal{C}}(X,Y) \Rightarrow \exists \int_{BG} f : X \longrightarrow Y$$

$\hookrightarrow$ sums of finitely many maps $X \to Y$ $\Rightarrow$ CAlg-enriched

Now: Can sum families of maps parametrized by spaces which aren't discrete.

In general:

$$\begin{pmatrix} \int \text{ indexed by} \\ (m-1)\text{-finite spaces} \end{pmatrix} \overset{\text{construct}}{\underset{(1)}{\rightsquigarrow}} \begin{pmatrix} \text{Norm maps between (co)limits} \\ \text{indexed by } m\text{-finite spaces} \end{pmatrix}$$

$$\begin{pmatrix} m\text{-Norm maps} \\ \text{being isom's} \end{pmatrix} \overset{\text{construct}}{\underset{(2)}{\rightsquigarrow}} \begin{pmatrix} \int \text{ indexed by} \\ m\text{-finite spaces} \end{pmatrix}$$

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(1) Given m-finite space A &

a functor $F : A \to \mathcal{C}$,

Want: $\mathrm{Nm}_A : \operatorname{colim}_A F \longrightarrow \lim_A F$

Need: $\mathrm{Nm}_A(a,b) : F(a) \longrightarrow F(b) \quad \forall\, a,b \in A$

$$F(a,b) : \underbrace{\mathrm{Map}_A(a,b)}_{(m-1)\text{-finite}} \longrightarrow \mathrm{Map}_{\mathcal{C}}(F(a), F(b))$$

$$\Rightarrow \mathrm{Nm}_A(a,b) := \int_{\mathrm{Map}_A(a,b)} F(a,b).$$

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(2) Given m-finite space A &

a map $f : A \to \mathrm{Map}_{\mathcal{C}}(X,Y)$,

Want: $\int_A f : X \longrightarrow Y$

$$X \longrightarrow \lim_A X \xrightarrow{f} \lim_A Y \xrightarrow{\mathrm{Nm}_A^{-1}} \operatorname{colim}_A Y \longrightarrow Y$$

Def: An $(m-1)$-semiadditive category $\mathcal{C}$ is <u>m-semiadditive</u> if $\forall$ m-finite A

$$\mathrm{Nm}_A : \operatorname{colim}_A \xrightarrow{\sim} \lim_A \quad \text{iso.}$$

<u>Consequences</u>

- Can integrate over m-finite spaces.

  $\Rightarrow$ enriched in "higher commutative monoids".
- <u>case</u>: $A \xrightarrow{\mathrm{id}_X} \mathrm{Map}_{\mathcal{C}}(X,X)$

  $\Rightarrow |A| := \int_A \mathrm{id}_X : X \to X$.

Def: A category $\mathcal{C}$ is <u>∞-semiadditive</u> if $\mathcal{C}$ is m-semiadditive $\forall m \geq -1$.

$\Rightarrow$ • limits & colimits agree over all π-finite spaces.
• can integrate over all π-finite spaces.

Thm: [Hopkins-Lurie] $\mathrm{Sp}_{K(n)}$ is ∞-semiadditive.

Thm: [CSY] $\mathrm{Sp}_{T(n)}$ is ∞-semiadditive.

combining two [CSY] results above ?

Thm: [CSY] R = ring spectrum.

Sp_R 1-semiadditive $\Rightarrow$ ∞-semiadditive.

↳ conjecture true $\forall$ presentable, stable ∞-cats.

Sketch that $Sp_{T(n)}$ is ∞-semiadditive.

The actual proof is written more generally.

Induction on semiadditivity level $m \geq 1$.

Assume $Sp_{T(n)}$ is m-semiadditive.

Want: $\forall B$ $(m+1)$-finite

[(0) $Nm_B : \operatorname{colim}_B \xrightarrow{\sim} \lim_B$.

Sequence of reductions of ↑

- Eilenberg-MacLane spaces sufficient
- p-local $\Rightarrow$ C_p sufficient.

[(1) Nm_B iso for $B = B^{m+1}C_p = K(C_p, m+1)$.

- given fiber sequence $A \to E \to B$

  m-finite

  $|A| : \mathbb{S}_{T(n)} \to \mathbb{S}_{T(n)}$ invertible $\Rightarrow$ Nm_B iso.

- if $\pi_k A \begin{cases} = 0, & k=0,\ k>m \\ \neq 0, & k=m \\ p\text{-group}, & \forall k \end{cases}$ A is m-good

  then $\exists$ fiber sequence $A \to E \to B$

[(2) $\exists$ m-good A s.t. $|A| \in \pi_0 \mathbb{S}_{T(n)}$ invertible.

- E_n = Morava E-theory $\Rightarrow$ sym. mon.

  $E_n \otimes (-) : Sp_{T(n)} \longrightarrow \widehat{\operatorname{Mod}}_{E_n}$ = K(n)-local E_n-modules

$\Rightarrow$ ring map $f: \pi_0 \mathbb{S}_{T(n)} \longrightarrow \pi_0 E_n = \mathbb{Z}_p[[u_1, \dots, u_{p-1}]]$

- $a \in \pi_0 \mathbb{S}_{T(n)}$ inv. $\Longleftrightarrow$ $f(a)$ inv.
- $\widehat{\mathrm{Mod}}_{E_n}$ also m-semiadditive & $f(|A|) = |A| \in \pi_0 E$

[(3) $\exists$ m-good A s.t. $|A| \in \pi_0 E$ invertible.

- [Bobkova-Goerss] $\mathrm{Im}(f) \subseteq \mathbb{Z}_p$

  invertible $\in \mathbb{Z}_p$ $\Longleftrightarrow$ p-adic valuation $= 0$.

- can reduce p-adic valuation of <u>non-zero</u> elements via Fermat quotient operation

$$\tilde{\delta}(x) = \frac{x - x^p}{p}$$

- show $\tilde{\delta}(|A|) = |A'| - |A''|$

  m-good

  $\Rightarrow$ replace A w/ A' or A''.

- $B^m C_p$ is m-good.

[(4) $0 \neq |B^m C_p| \in \pi_0 E_n$.

- $A \otimes E_n := \operatorname{colim}_A E_n \in \widehat{\mathrm{Mod}}_{E_n}$.

  $\Rightarrow$ dualizable &

  $\dim(A \otimes E_n) = |A^{S^1}| \in \pi_0 E_n$.

- $|(B^m C_p)^{S^1}| = |B^m C_p \times B^{m-1} C_p| = |B^m C_p| |B^{m-1} C_p|$?

[(5) $0 \neq \dim(B^m C_p \otimes E_n) \in \pi_0 E_n$.

- $\dim(A \otimes E_n) = \chi_{K(n)}(A)$
$$= \dim_{\mathbb{F}_p} K(n)_0 A - \dim_{\mathbb{F}_p} K(n)_1 A$$

[(6) $0 \neq \chi_n(B^m C_p) \in \mathbb{Z}$.

↳ computed by [Ravenel-Wilson]. //

More characterizations of higher semi-$\oplus$ in Sp

$R \neq 0$ p-local ring spectrum. TFAE:

- Sp_R is 1-semiadditive.
- Sp_R is ∞-semiadditive.
- $\exists n \geq 0 \mid Sp_{K(n)} \subseteq Sp_R \subseteq Sp_{T(n)}$.
- $\exists! n \geq 0 \mid R \otimes K(n) \neq 0$ & $R \otimes H\mathbb{F}_p = 0$.
- $Sp_R = Sp_{H\mathbb{Q}}$ or $Sp_R \xrightarrow{\Omega^\infty} S_*$ has a retract.

Application of ∞-semiadditivity – alternate char's of chromatic height.

R = p-local ring spec. $d \geq 0$.

TFAE:

- $R \otimes K(m) = 0 \quad \forall m > d$
- $R \otimes F(d+1) = 0$ for some finite spec. $F(d+1)$ type $d+1$.
- $R \otimes \Sigma^\infty A = 0 \quad \forall$ d-connected π-finite space A.