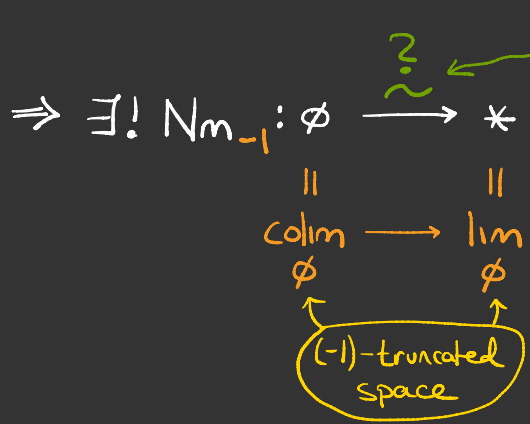


Higher Semicadditivity (Talk notes, 1/24/24)

\mathcal{C} = Category w/ initial \emptyset & terminal $*$.

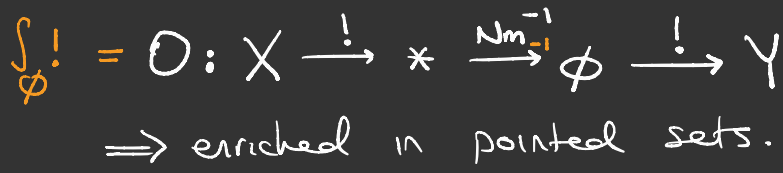


pointed

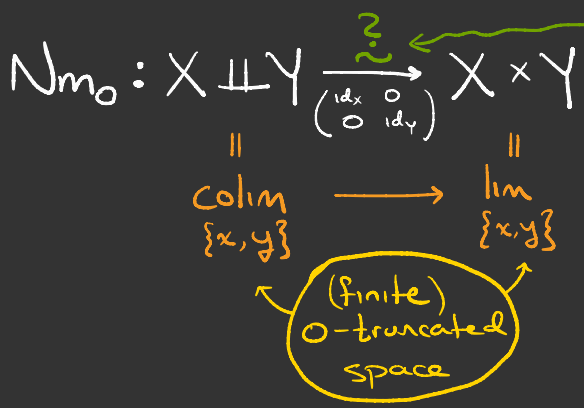
yes	no
Group	Set
Ab	S
Vect _k	
S*	
Sp	
all stable ∞ -cats	

Consequences of pointedness

$\forall X, Y \in \mathcal{C}$ i.e. $\emptyset \rightarrow \text{Map}_{\mathcal{C}}(X, Y)$ \exists zero map



\bullet if \mathcal{C} has finite (co)products: $\forall X, Y \in \mathcal{C}, \exists$



Semiadditive

yes	no
Ab	Set
Vect _k	S
Sp	Group
all stable ∞ -cats	S*

Consequences of semiadditivity (repeat above)

- (use Nm_0^{-1}) $\forall f, g: X \rightarrow Y \quad \exists f+g$

$$X \xrightarrow{f+g} Y \times Y \xrightarrow{Nm_0^{-1}} Y \amalg Y \xrightarrow{id \times id} Y$$

$f+g$

Further, since $\prod_{i=1}^n X_i \xrightarrow{\sim} \prod_{i=1}^n X_i$

$$\Rightarrow \left(\underbrace{f_1, \dots, f_n : X \rightarrow Y}_{\{1, \dots, n\} \rightarrow \text{Map}_c(X, Y)} \Rightarrow \exists \int_{\{1, \dots, n\}}^{\amalg} f_i : X \rightarrow Y \right)$$

\Rightarrow enriched in $\text{CAlg}(S)$

\Rightarrow multiplication by n :

$$\underbrace{id_X, \dots, id_X : X \rightarrow X}_{n = |\{1, \dots, n\}|} \Rightarrow n \cdot \int_{\{1, \dots, n\}}^{\amalg} id_X : X \rightarrow X$$

Def: $m \geq -1$. A space A is m -finite if

$$\pi_k A = \begin{cases} \text{finite}, & k \leq m \\ 0, & k > m \end{cases}$$

A space A is π -finite if A is m -finite for some m .

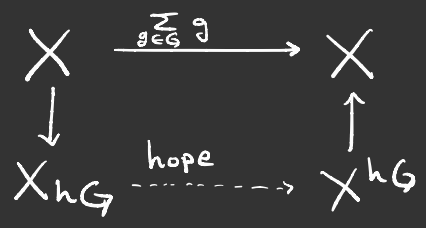
• 1-finite space $\rightarrow BG, G = \text{finite group}$.

$$\left(\begin{array}{c} \text{functor} \\ BG \xrightarrow{F} \mathcal{C} \end{array} \right) \Leftrightarrow (G \subset X \in \mathcal{C}) \quad w/$$

$$\text{colim}_{BG} F \simeq X_{hG} \quad \& \quad \lim_{BG} F \simeq X^{hG}$$

Want: $Nm_i: X_{hG} \rightarrow X^{hG}$

Roughly,



Ex: (1-cat'l) $\mathcal{C} = Ab \ni A \supset G$

$\Rightarrow \sum_{g \in G} g$ well-defined on orbits & image $\in AG$

$\Rightarrow Nm_i: A_G \xrightarrow{|G|^{-1}} A^G \rightsquigarrow$ Not usually iso!

e.g. $C_2 \subset \mathbb{Z}/2$ trivial $\Rightarrow Nm_i(x) = \sum_{g \in C_2} x = 2x = 0$.

if $\mathcal{C} = \text{Vect}$, then $Nm_i: V_G \simeq V^G$.

Def: A (0-)semiadditive category \mathcal{C} is 1-semiadditive if $Nm_i: X_{hG} \xrightarrow{\sim} X^{hG}$ iso \forall finite $G \subset X \in \mathcal{C}$ (any 1-finite space is $\simeq \coprod_{i=1}^n BG_i$)

<u>1-semiadditive</u>	
yes	no
Vect @ SP _{HQ}	most others.

Within Sp:
1-semiadditivity is closely related with chromatic phenomena.

Thm:

- (1) [Greenlees-Hovey-Sadofsky] $SP_{K(n)}$ is 1-semiadditive.
- (2) [Kuhn] $SP_{T(n)}$ is 1-semiadditive.

$\hookrightarrow Nm_1: X_{hG} \rightarrow X^{hG} \text{ iso}$

$\Leftrightarrow \text{cof}(N_1) \simeq \underline{X^{tG}} \simeq 0.$

Thm: [Carmeli-Schlank-Yanovski] $0 \neq \mathbb{R}$ p-local homotopy ring spectrum.

$$\left(\begin{array}{l} SP_{\mathbb{R}} \text{ is} \\ 1\text{-semiadditive} \end{array} \right) \Leftrightarrow \left(\begin{array}{l} \exists n \geq 0 \text{ s.t.} \\ SP_{K(n)} \subseteq SP_{\mathbb{R}} \subseteq SP_{T(n)} \end{array} \right)$$

Consequences of 1-semiadditivity

$B\mathbb{G} \xrightarrow{f} \text{Map}_e(X, Y) \Rightarrow \exists \int_{B\mathbb{G}} f : X \rightarrow Y$

\hookrightarrow sums of finitely many maps $X \rightarrow Y \Rightarrow \text{Cat}_{\mathbb{G}}$ -enriched

Now: can sum families of maps parametrized by spaces which aren't discrete.

In general:

$\left(\int \text{ indexed by } (m-1)\text{-finite spaces} \right) \xrightarrow[\text{(1)}]{\text{construct}} \left(\text{Norm maps between } (m)\text{ limits indexed by } m\text{-finite spaces} \right)$

$\left(m\text{-Norm maps being isom's} \right) \xrightarrow[\text{(2)}]{\text{construct}} \left(\int \text{ indexed by } m\text{-finite spaces} \right)$

(1) Given m -finite space A & a functor $F: A \rightarrow \mathcal{C}$,

want: $Nm_A: \text{colim}_A F \rightarrow \text{lim}_A F$

Need: $Nm_A(a,b): F(a) \rightarrow F(b) \quad \forall a,b \in A$

$F(a,b): \text{Map}_A(a,b) \rightarrow \text{Map}_{\mathcal{C}}(F(a), F(b))$

\uparrow $(m-1)$ -finite

$\Rightarrow Nm_A(a,b) := \int_{\text{Map}_A(a,b)} F(a,b)$

(2) Given m -finite space A & a map $f: A \rightarrow \text{Map}_{\mathcal{C}}(X, Y)$,

want: $\int_A f: X \rightarrow Y$

$X \rightarrow \text{lim}_A X \xrightarrow{f} \text{lim}_A Y \xrightarrow{Nm_A^{-1}} \text{colim}_A Y \rightarrow Y$

Def: An $(m-1)$ -semiadditive category \mathcal{C} is m -semiadditive if $\forall m$ -finite A

$$Nm_A : \operatorname{colim}_A \xrightarrow{\sim} \operatorname{lim}_A \quad \text{iso.}$$

Consequences

- Can integrate over m -finite spaces.
 \Rightarrow enriched in "higher commutative monoids".
- case: $A \xrightarrow{\operatorname{id}_X} \operatorname{Map}_{\mathcal{C}}(X, X)$
 $\Rightarrow |A| := \int_A \operatorname{id}_X : X \rightarrow X.$

Def: A category \mathcal{C} is ∞ -semiadditive if \mathcal{C} is m -semiadditive $\forall m \geq -1$.

- \Rightarrow
- limits & colimits agree over all π -finite spaces.
 - can integrate over all π -finite spaces.

Thm: [Hopkins-Lurie] $\operatorname{Sp}_{\mathbb{K}(c_n)}$ is ∞ -semiadditive.

Thm: [CSY] $\operatorname{Sp}_{\mathbb{T}(c_n)}$ is ∞ -semiadditive.
 combining two [CSY] results above \supseteq

Thm: [CSY] $\mathbb{R} = \text{ring spectrum}$.

$\operatorname{Sp}_{\mathbb{R}}$ 1-semiadditive $\Rightarrow \infty$ -semiadditive.

\hookrightarrow conjecture true \forall presentable, stable ∞ -cats.

Sketch that $\mathcal{S}p_{T(n)}$ is ∞ -semiadditive.

The actual proof is written more generally.

Induction on semiadditivity level $m \geq 1$.

Assume $\mathcal{S}p_{T(n)}$ is m -semiadditive.

Want: $\forall B$ $(m+1)$ -finite

$$[(0) \quad Nm_B : \text{colim}_B \xrightarrow{\sim} \text{lim}_B]$$

Sequence of reductions of \rightarrow

- Eilenberg-MacLane spaces sufficient
- p -local $\Rightarrow C_p$ sufficient.

$$[(1) \quad Nm_B \text{ iso for } B = B^{m+1}C_p = k\langle C_p, m+1 \rangle.]$$

- given fiber sequence $A \rightarrow E \rightarrow B$
 $\uparrow \quad \quad \uparrow$
 $m\text{-finite}$

$$|A| : \mathcal{S}p_{T(n)} \rightarrow \mathcal{S}p_{T(n)} \text{ invertible} \Rightarrow Nm_B \text{ iso.}$$

- if $\pi_k A \begin{cases} = 0, & k=0, k > m \\ \neq 0, & k=m \\ p\text{-group}, & \forall k \end{cases} \left. \vphantom{\pi_k A} \right] \underline{A \text{ is } m\text{-good}}$

then \exists fiber sequence $A \rightarrow E \rightarrow B$

$$[(2) \quad \exists m\text{-good } A \text{ s.t. } |A| \in \pi_0 \mathcal{S}p_{T(n)} \text{ invertible.}]$$

- $E_n = \text{Morava } E\text{-theory} \Rightarrow \text{sym. mon.}$

$$E_n \otimes (-) : \mathcal{S}p_{T(n)} \rightarrow \widehat{\text{Mod}}_{E_n} = \begin{matrix} k(n)\text{-local} \\ E_n \text{ modules} \end{matrix}$$

18/9
⇒ ring map $f: \pi_0 \mathcal{S}_{T(n)} \rightarrow \pi_0 E_n = \mathbb{Z}_p \llbracket u_1, \dots, u_{p-1} \rrbracket$

• $a \in \pi_0 \mathcal{S}_{T(n)}$ inv. $\Leftrightarrow f(a)$ inv.

• $\widehat{\text{Mod}}_{E_n}$ also m -semiadditive & $f(|A|) = |A| \in \pi_0 E$

[(3)] \exists m -good A s.t. $|A| \in \pi_0 E$ invertible.

• [Bokawa-Goerss] $\text{Im}(f) \subseteq \mathbb{Z}_p$

invertible $\in \mathbb{Z}_p \Leftrightarrow p$ -adic valuation = 0.

• can reduce p -adic valuation of non-zero elements via Fermat quotient operation

$$\tilde{f}(x) = \frac{x - x^p}{p}$$

• show $\tilde{f}(|A|) = |A'| - |A''|$

⇒ replace A w/ A' or A'' .

• $B^m C_p$ is m -good.

[(4)] $0 \neq |B^m C_p| \in \pi_0 E_n$

• $A \otimes E_n := \text{colim}_A E_n \in \widehat{\text{Mod}}_{E_n}$

⇒ dualizable &

$$\dim(A \otimes E_n) = |A^S| \in \pi_0 E_n$$

• $|(B^m C_p)^S| = |B^m C_p \times B^{m-1} C_p| = |B^m C_p| |B^{m-1} C_p|$?

[(5)] $0 \neq \dim(B^m C_p \otimes E_n) \in \pi_0 E_n$

- $$\dim(A \otimes E_n) = \chi_{K(n)}(A)$$

$$= \dim_{\mathbb{F}_p} K(n)_* A - \dim_{\mathbb{F}_p} K(n)_! A$$

[(6) $0 \neq \chi_n(B^m \mathbb{C}_p) \in \mathbb{Z}$.

↳ computed by [Ravenel-Wilson] . //

More characterizations of higher semi- \otimes in Sp

$R \neq 0$ p -local ring spectrum. TFAE!

- Sp_R is 1-semiadditive .
- Sp_R is ∞ -semiadditive .
- $\exists n \geq 0 \mid Sp_{K(n)} \subseteq Sp_R \subseteq Sp_{T(n)}$.
- $\exists ! n \geq 0 \mid R \otimes K(n) \neq 0$ & $R \otimes HF_p = 0$.
- $Sp_R = Sp_{H\mathbb{O}}$ or $Sp_R \xrightarrow{\Omega^\infty} S_*$ has a retract .

Application of ∞ -semiadditivity - alternate char's of chromatic height . $R = p$ -local ring spec. $d \geq 0$.

TFAE!

- $R \otimes K(m) = 0 \quad \forall m > d$
- $R \otimes F(d+1) = 0$ for some finite spec. $F(d+1)$ type $d+1$.
- $R \otimes \Sigma^\infty A = 0 \quad \forall d$ -connected π -finite space A .