# The Homotopy Fixed Point Spectral Sequence for $KU^{hC_2}$

## I. Stable Homotopy Theory

space $X \rightsquigarrow$ "$\mathbb{R} \times X$" ~ like $X$ but one dimension larger

homotopically meaningfull: $\Sigma X := S^1 \wedge X$ = the <u>suspension</u> of $X$

"Stable" phenomena: properties of $X$ that only depend on $\Sigma^n X$ for $n \gg 0$.

$\subseteq Sp$ = homotopy theory of <u>spectra</u>.

<u>Ex:</u> $\pi_n^s(X) = \operatorname{colim}_k \pi_{n+k} \Sigma^k X = \operatorname{colim}_k [S^{n+k}, S^k \wedge X]$

↳ actually defined on all objects in $Sp$

will denote $\pi_n$.

A <u>cohomology theory</u> is a (homotopy invariant) functor

$$E^* : Top_*^{op} \longrightarrow Ab_{\mathbb{Z}}$$

satisfying the "Eilenberg-Steenrod axioms."

↳ come with tools that make them computable invariants of spaces.

<u>Thm (Brown)</u>: Every cohomology theory is represented by a spectrum.

In fact,

$$\begin{pmatrix}\text{cohomology}\\ \text{theories}\end{pmatrix} \simeq (\text{spectra})$$

$\Rightarrow$ cohom. theories have homotopy groups

& $\pi_* E = E^{-*}(pt)$

<u>Ex:</u> (Ordinary Cohomology) For an abelian group $A$, $\exists$ $HA$ characterized by

$$\pi_* HA = \begin{cases} A, & * = 0 \\ 0, & \text{else} \end{cases}$$

& for a (finite) group $G$,

$$HA^*(BG) \cong H^*(G, A).$$

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## II. Topological K-theory

Cohomology theories determined by linear algebra parametrized by spaces.

$$KU^0(X) = Gr(\{\mathbb{C}\text{-vector bundles on } X\}/\cong)$$

$$KO^0(X) = Gr(\{\mathbb{R}\text{-vector bundles on } X\}/\cong)$$

↳ actually multiplicative via $\otimes$

$\Rightarrow$ $\pi_* KU$ & $\pi_* KO$ are graded rings.

<u>Fact</u>:

dimension of vector spaces

$$\pi_* KU = \begin{cases} \mathbb{Z}, & * = 0 \ \boxed{\text{mod } 2} \\ 0, & * = 1 \ \boxed{\text{mod } 2} \end{cases}$$

Bott Periodicity

$GL_n(\mathbb{C})$ is connected

↳ ring structure generated by $u \in \pi_2 KU$.

↳ Bott Periodicity for KO is 8-periodic. need more.

- $C_2 \circlearrowright KU$ s.t. $C_2 \circlearrowright \pi_* KU$ is given by $u \mapsto -u$

- $\left\{ \begin{matrix} \text{spectra with} \\ G\text{-action} \end{matrix} \right\} \longrightarrow Sp_G \simeq$ Cohomology theories as $G$-spaces.

  $C_2 \circlearrowright KU \longmapsto KR \approx$ "Real" vector bundles

- $C_2 \circlearrowright X$ trivially, $KR(X) \cong KO(X)$

  $\Rightarrow KU^{hC_2} \simeq KO$

## III. The HFPSS

Given a spectrum $X$ with $G$-action

$\exists$ spectral sequence (HFPSS) with

$$E_2^{p,q} = H^p(G, \pi_q X) \Rightarrow \pi_{q-p}(X^{hG})$$

↑ as a $G$-module, since $\pi_q(G \circlearrowright X) = G \circlearrowright \pi_q X$.

& $|d_r| = (r, r-1)$.

[**Goal:** Compute $\pi_* KO$ via HFPSS for $C_2 \circlearrowright KU$

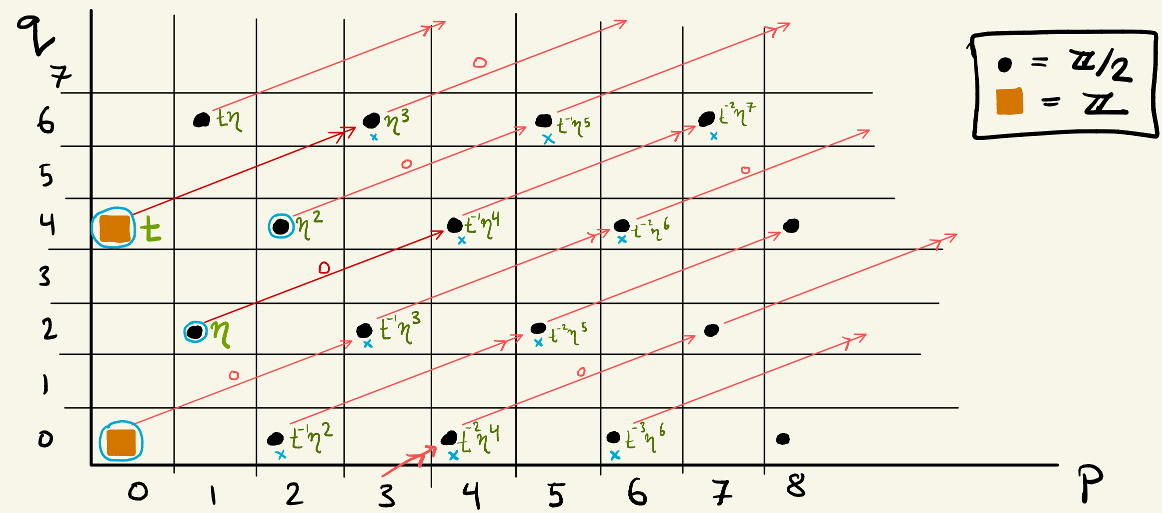
Need $H^p(C_2, \pi_q KU)$.

$$\pi_q KU = \begin{cases} \mathbb{Z}, & q = 0 \mod 4 \\ \mathbb{Z}_\sigma, & q = 2 \mod 4 \\ 0, & q \text{ odd} \end{cases}$$

$$H^p(C_2, \mathbb{Z}) = H\mathbb{Z}^p(\mathbb{RP}^\infty) = \begin{cases} \mathbb{Z}, & p = 0 \\ \mathbb{Z}/2, & p > 0 \text{ even} \\ 0, & p \text{ odd} \end{cases}$$

$$H^p(C_2, \mathbb{Z}_\sigma) = \begin{cases} \mathbb{Z}/2, & p \text{ odd} \\ 0, & p \text{ even} \end{cases}$$

Putting together we get $E_2^{p,q} =$




$|d_2| = (2,1) \implies = 0$

$|d_3| = (3,2)$ potentially nonzero.

- ring structure: $\mathbb{Z}[t^{\pm}, \eta]/(2\eta)$
- "Bott element": $\mathbb{CP}^1 \longrightarrow KU \implies$ map of HFPSS
  $S^2$ $\implies d_3(\eta) = 0$
- linear algebra $\implies \pi_0 KO \cong \mathbb{Z} \implies (4,4)$-component must vanish $\implies d_3(t^{-1}\eta^4) \neq 0$
  $\implies d_3(t) = \eta^3$ or else $d_3 \equiv 0$.

- $d(t^{2k}) = d(t^k)t^k \pm t^k d(t^k) = 0 \quad , \quad d(\eta^{2k}) = 0$

$$d(t^{2k+1}) = d(t^{2k})t \pm t^{2k} d(t) = t^{2k}\eta^3$$

$$d(\eta^{2k+1}) = \eta^{2k} d(\eta) = 0$$

$$d(t^k\eta^\ell) = d(t^k)\eta^\ell + t^k \cancelto{0}{d(\eta^\ell)}$$

$$= \begin{cases} 0, & k \text{ even} \\ t^{k-1}\eta^{\ell+3}, & k \text{ odd} \end{cases}$$

i.e.

$$d = \begin{cases} \text{surj.}, & \text{odd powers of } t \\ 0, & \text{else} . \end{cases}$$

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<u>whats left</u>

| $q-p$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\pi_{q-p}KO$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |

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