THE DERIVATIVE OF arctan(x)

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GOAL: Compute $\frac{d}{dx}\arctan(x)$ $\left[=\arctan'(x)\right]$.

① "Trick": Set up a simple equation involving $\arctan x$.

arctan(x) = the inverse function of tan(x) $\rightsquigarrow$ $\tan(\arctan x) = x$.

② Differentiate both sides of the equation.

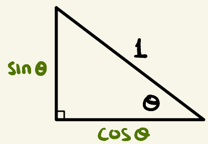
$$\frac{d}{dx}\Big[\tan(\arctan(x))\Big] = \frac{d}{dx}x$$

chain rule & $\tan' x = \sec^2 x$

$$\sec^2(\arctan x)\,\arctan'(x) = 1 \overset{\sec x = \frac{1}{\cos x}}{\rightsquigarrow} \boxed{\arctan'(x) = \cos^2(\arctan x)} \quad ★$$

↳ need a better expression for "$\cos^2(\arctan x)$"...

③ Draw a triangle to reinterpret trig expressions.



= right triangle w/ hypotenuse = 1 & an angle θ.

Recall: "SOH-CAH-TOA" ⇒ base = cos θ & height = sin θ.

Since $\arctan\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right) \Rightarrow \theta = \arctan\left(\frac{\sin\theta}{\cos\theta}\right)$

$\Rightarrow \cos\theta = \cos\left(\arctan\left(\frac{\sin\theta}{\cos\theta}\right)\right)$. Since we want $\cos^2(\arctan x)$ ...

$$\boxed{\text{Let } x = \frac{\sin\theta}{\cos\theta}} \Rightarrow \boxed{\cos^2\theta = \cos^2(\arctan x)}.$$

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④ Convert expressions back in terms of "x".

"Pythagorean theorem" $\sin^2\theta + \cos^2\theta = 1$

$$x = \frac{\sin\theta}{\cos\theta} \Rightarrow x^2 = \frac{\sin^2\theta}{\cos^2\theta} \Rightarrow \cos^2\theta = \frac{\sin^2\theta}{x^2} = \frac{1-\cos^2\theta}{x^2}$$

$$\Rightarrow \frac{1}{x^2} = \cos^2\theta + \frac{\cos^2\theta}{x^2} = \cos^2\theta\left(1 + \frac{1}{x^2}\right)$$

by combining with ★ & ★★

$$\Rightarrow \cos^2\theta = \frac{1/x^2}{(1+1/x^2)} = \frac{1}{x^2(1+\frac{1}{x^2})} = \frac{1}{x^2+1} \Rightarrow \boxed{\arctan'(x) = \frac{1}{x^2+1}}$$

Note: The same ideas work to find $\arcsin'(x)$ & $\arccos'(x)$, and those turn out to be a bit easier.