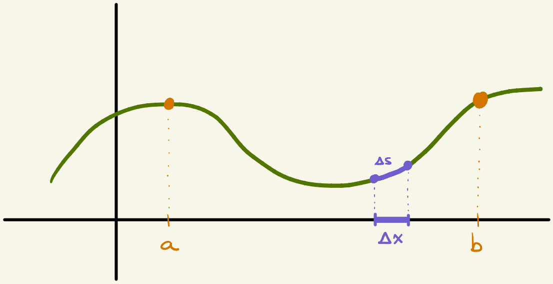
Arc Length I — Yigal Kamel

Want: Length of the curve defined by the graph of $f(x)$



Over a small interval Δx, we can try to use Δx to approximate the length, Δs, of the corresponding segment of the graph.

ISSUE: The steeper the slope of $f(x)$ on the interval of Δx, the more wrong this approximation will be — this error does **NOT** vanish in the limit!

↳ [when the word "slope" appears, you know there will be a derivative!]

FIX: Pythagorean theorem $\Rightarrow (\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$

$$\Longrightarrow \Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x)^2\left[1 + \frac{(\Delta y)^2}{(\Delta x)^2}\right]} = \Delta x\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

Taking a limit as $\Delta x \to 0$, $\quad \frac{\Delta y}{\Delta x} = f'(x) \quad$ (by definition!)

$$\Longrightarrow \boxed{ds = \sqrt{1 + f'(x)^2}\, dx}\,.$$

Notice: When $f'(x) = 0 \rightsquigarrow ds = \sqrt{1}\, dx = dx$.
In other words, when the graph is horizontal (= no slope), our original "bad" approximation is exactly right!

Looking ahead: In a sense, this isn't a "natural" way to look at arc length of a curve. **Better:** View the curve as being "traced out over time" $\to (x(t), y(t))$ where each coordinate is a function of t.

Q: Can you formulate the arc length of such a curve as a function of t?