Twisted equivariant K-theory

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1 Motivation

Given a group G, the Atiyah-Segal completion theorem [3] provides a precise relationship between $K(BG)$ and $R(G)$. Namely,

$$K^*(BG) \cong R(G)_{\hat{I}_G}.$$

Since BG is a "delooping" of G, we can ask if $K^*(G)$ can be used to describe $R(LG)$, where LG is the *loop group* of smooth maps from the circle into G. Freed, Hopkins, and Teleman [7] realized this goal as an isomorphism

$$K_G^{\mathfrak{g}+\check{h}+\tau}(G) \cong R^\tau(LG). \tag{1}$$

The subscript refers to the action of G on itself by conjugation, and the superscript refers to a *twist* of equivariant K-theory. Our goal in this talk is to understand what twists of K-theory are and how the associated twisted (equivariant) K-theory groups are defined. In the process, we will also refine the category (previously RTop) that will serve as the domain for twisted K-theory.

2 Twists of cohomology theories

2.1 What does it mean to twist an object?

The following table is a perspective on the word "twist" motivated by the idea that a space is a twisted point.

object	twisted object
$*$	space X
vector space	vector bundle $V \to X$
group	principal bundle $P \to X$

Note that in the table above,

- twisted objects depend on the input of a specific space X to "twist over",
- the fibers of a twisted object are isomorphic to a specific untwisted object.

In this spirit, we can call a real (complex) vector bundle $V \to X$ of rank n a *twist* of $\mathbb{R}^n$ ($\mathbb{C}^n$) over X. Similarly, a principal G-bundle $P \to X$ can be thought of as a *twist* of G over X.

We want to define twists of a cohomology theory. As a test case, let's define twists of a ring. Looking more closely at the examples of vector bundles and groups, we can notice a couple things:

- principal bundles forget the identity elements of its fibers;
- twisted objects over X can be classified by maps into an appropriate classifying space.

The idea of "forgetting the identity" for rings amounts to considering bundles whose fibers are free R-modules of rank 1. We expand our table to include rings and the associated homotopical description in terms of classifying maps.

object	twisted object	classified by
$*$	space X	$X \to B\mathrm{Aut}(*) \simeq *$
vector space	vector bundle $V \to X$	$X \to BGL_n(k)$
group	principal bundle $P \to X$	$X \to BG$
ring	bundle of free R-modules of rank 1	$X \to BGL_1(R) \simeq BR^\times$

In order to get a handle on what the appropriate notion for cohomology is, let's consider a familiar example.

Example 2.1.1. Ordinary cohomology with local coefficients.
Given an abelian group A, we can define $H^*(X; A) =$ singular cohomology of X with coefficients in A. We can replace A with a local system of abelian groups, i.e. a presheaf

$\mathcal{A} : \Pi_{\leq 1}X \to \mathsf{Ab}$, and similarly define $H^*(X; \mathcal{A})$ = singular cohomology of X with coefficients in $\mathcal{A}$.

Given a real vector bundle of rank n, $\pi : V \to X$, define $\mathcal{A}_V(p) = \tilde{H}^n(\pi^{-1}(p); \mathbb{Z}) \cong \mathbb{Z}$. Then there is a "twisted" Thom isomorphism

$$H^*(X; \mathcal{A}_V) \cong H^{*+n}(X^V; \mathbb{Z}).$$

An orientation of V amounts to a natural isomorphism of functors $\mathcal{A}_V \cong \mathbb{Z}$, which induces the untwisted Thom isomorphism $H^*(X; \mathbb{Z}) \cong H^{*+n}(X^V; \mathbb{Z})$.

Remark 2.1.2. The approach in Example 2.1.1 is fine, but in the spirit of the perspective above on twisted objects, we want to realize these and more general twists of cohomology as "bundles of spectra". From this homotopical point of view, requiring a path to induce an isomorphism is too strong, so we need to replace the fundamental groupoid $\Pi_{\leq 1}X$ with the fundamental ∞-groupoid = singular complex of X: $\Pi_\infty X = \mathrm{Sing} X$.

Following the idea of twisted rings as bundles of free rank 1 modules, we make the following definitions, using Example 2.1.1 as a prototype.

Definition 2.1.3 ([1])**.** Let A be a cohomology theory (by which we mean S-*algebra*, where S is the sphere spectrum; see [1]).

- $\mathrm{Mod}_A :=$ ∞-category of A-modules;
- $\mathrm{Line}_A :=$ subcategory of Mod_A consisting of the A-*lines*, i.e. A-modules that are weakly equivalent to A.
- A *twist* of A over a space X is a functor $\mathrm{Sing} X \to \mathrm{Line}_A$.

It turns out that the geometric realization of Line_A is a familiar space:

$$|\mathrm{Line}_A| \simeq BGL_1 A,$$

where $GL_1 A$ is the subspace of $\Omega^\infty A$ consisting of the connected components that are units in the ring $\pi_0(\Omega^\infty A)$:

$$\begin{tikzcd} GL_1 A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Omega^\infty A \arrow[d] \\ \pi_0(\Omega^\infty A)^\times \arrow[r] & \pi_0(\Omega^\infty A). \end{tikzcd} \tag{2}$$

Thus, twists of a cohomology theory A are classified by maps (of spaces) $X \to BGL_1 A$. We can now add cohomology to the table:

object	twisted object	classified by
$*$	space X	$X \to B\mathrm{Aut}(*) \simeq *$
vector space	vector bundle $V \to X$	$X \to BGL_n(k)$
group	principal bundle $P \to X$	$X \to BG$
ring	bundle of free R-modules of rank 1	$X \to BGL_1(R) \simeq BR^\times$
cohomology	bundle of A-modules weakly equivalent to A	$X \to BGL_1 A$

2.2 Twisted cohomology.

While we could have started with the space-level classification of twists of cohomology, the infinity-categorical language leads to a convenient definition of the corresponding twisted cohomology of a space.

Definition 2.2.1 ([1])**.** Given a twist $f : \mathrm{Sing}X \to \mathrm{Line}_A$,

- the *Thom spectrum* of f is

$$X^f := \mathrm{colim}(\mathrm{Sing}X \xrightarrow{f} \mathrm{Line}_A \subset \mathrm{Mod}_A) \; = \text{“}\operatorname*{colim}_{p \in X} f(p)\text{”} \in \mathrm{Mod}_A;$$

- the *f-twisted A-cohomology* of X (in degree n) is

$$A^{n+f}(X) := [X^f, \Sigma^n A]_{\mathrm{Mod}_A}.$$

Example 2.2.2. Cohomology twisted by vector bundles.
Given a vector bundle V classified by a map $X \to BO$, we obtain a twist as in Definition 2.1.3 (which we also denote by V) as follows:

$$\begin{tikzcd} X \arrow[r] \arrow[ddr, "V"'] & BO \arrow[d, "BJ"] \\ & BGL_1 S \arrow[d, "- \wedge A"] \\ & BGL_1 A \end{tikzcd}$$

where BJ is induced by the J-homomorphism, and $- \wedge A$ is “change of scalars” given by tensoring with A. Then the V-twisted A-cohomology of X is

$$\begin{aligned} A^{*+V}(X) &= [X^V \wedge A, \Sigma^* A]_{\mathrm{Mod}_A} \\ &\cong [X^V, \Sigma^* A]_{\mathrm{Mod}_S} \\ &= A^*(X^V), \end{aligned}$$

so we recover the Thom isomorphism from Example 2.1.1. In fact, this suggests that we can view the definition of twisted cohomology for general twists as a generalized twisted Thom isomorphism.

Remark 2.2.3. In Example 2.2.2 we abused notation and identified two different notions of Thom spectrum. This is justified by Theorem 4.5 in [1], which is proved in [2]. See [1] and [2] for a more extensive discussion of Thom isomorphisms and orientations.

Example 2.2.4. General twists of ordinary cohomology.
We know vector bundles twist ordinary cohomology. We can ask if there are any other twists. Let $H\mathbb{Z}$ be the Eilenberg-Mac Lane spectrum with integer coefficients. Then (2) becomes

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z} \arrow[d] \\ \mathbb{Z}/2 \arrow[r] & \mathbb{Z}, \end{tikzcd}$$

So $BGL_1(H\mathbb{Z}) \simeq B(\mathbb{Z}/2) \simeq \mathbb{RP}^\infty$, which classifies real *line* bundles. Thus,

$$\{\text{twists} : X \to BGL_1(H\mathbb{Z})\} \cong \{\text{real line bundles on X}\}.$$

This shows that not all vector bundles yield *distinct* twists of cohomology; only the orientation line bundle of a vector bundle contributes to the twist. Another way to say the same thing is that the suspension isomorphism (i.e. cohomological degree) is not considered a twist from this point of view. We can, in fact, incorporate degree into this general framework for twists by replacing Line_A with $\mathrm{Pic}(A) = \mathrm{Pic}(\mathrm{Mod}_A)$. Roughly speaking, this replaces "free module of rank 1" with "invertible module". We'll return to this in section 3.4.

3 Twists of K-theory

3.1 General twists.

As in Definition 2.1.3, a twist of K-theory, is a map $X \to BGL_1K$, where K is the (complex) K-theory spectrum [8].

Proposition 3.1.1 (stated in [6])**.** $GL_1K \simeq \mathbb{Z}/2 \times \mathbb{CP}^\infty \times BSU$.

After delooping:

- the $\mathbb{Z}/2$ factor classifies twists $\tau \in [X, B\,\mathbb{Z}/2] \cong H^1(X; \mathbb{Z}/2)$, which are the same as the twists arising from line bundles in Example 2.2.4 (since $B\,\mathbb{Z}/2 \simeq K(\mathbb{Z}/2, 1)$).
- For the $\mathbb{CP}^\infty$ factor, recall that $\mathbb{CP}^\infty \simeq BU(1)$ (since it is a classifying space for complex line bundles), and $U(1) \simeq B\,\mathbb{Z}$. Thus, the corresponding twists of K-theory are
$$\tau \in [X, B\,\mathbb{CP}^\infty] \cong [X, B^3\,\mathbb{Z}] \cong [X, K(\mathbb{Z}, 3)] \cong H^3(X; \mathbb{Z}).$$
- We will ignore the twists arising from BSU in this talk.

3.2 The Atiyah-Segal model for twists in $H^3(X;\mathbb{Z})$.

We are interested in geometric representatives for twists of K-theory. There are a number of ways to achieve this; the one that seems closest to their characterization above is presented in [4].

Consider the unitary group $U(\mathcal{H})$ of a separable Hilbert space $\mathcal{H}$. Atiyah and Segal show that $U(\mathcal{H})$ is contractible, and it has an evident free action by $U(1)$. Thus, $U(\mathcal{H})$ with this $U(1)$-action is a model for the universal bundle $EU(1)$. In particular,

$$PU := U(\mathcal{H})/U(1) \simeq BU(1).$$

By the discussion above, we see that

$$\begin{aligned}\{\text{twists } \tau \in H^3(X;\mathbb{Z})\} &\cong [X, BPU] \\ &\cong \{PU\text{-bundles over } X\} \\ &\cong \{P(\mathcal{H})\text{-bundles over } X\}.\end{aligned}$$

So we can view a projective Hilbert bundle as a way to twist K-theory over a space.

Remark 3.2.1. As stated, projective Hilbert bundles correspond to twists in $H^3(X;\mathbb{Z})$.

- We can also recover the twists that correspond to elements of $H^1(X;\mathbb{Z}/2)$ by adding a $\mathbb{Z}/2$-grading to the Hilbert bundles.
- It is often convenient to use finite dimensional geometric models for homotopical objects. To this end, we can ask which twists arise from considering projective Hilbert bundles with finite dimensional $\mathcal{H}$. It turns out that such bundles correspond exactly to the torsion elements in $H^3(X;\mathbb{Z})$; see [4].

Thus, in Atiyah and Segal's model for twists of K-theory, infinite dimensional geometry is unavoidable. We can circumvent the infinite dimensionality by "raising the category level" of our "bundles". This is the model presented by Freed in [5]. In the process, Freed also realizes cohomological degree as a twist, i.e. he constructs a finite dimensional model for $\mathrm{Pic}(\mathrm{Mod}_A)[0,3]$ (see Example 2.2.4). In order to (cleanly) describe what such "higher categorical bundles" are, we should reexamine what we mean by spaces.

3.3 Local quotient groupoids (topological stacks).

$$
\begin{tikzcd}
 & \text{“spaces” } X \arrow[dl, dashed, "\text{want}"'] \arrow[dr, dashed, "\text{want}"] & \\
\text{twisted \emph{equivariant}} \\ K\text{-theory} \arrow[d] & & \text{use open covers to define} \\ \text{bundles/local systems} \arrow[d] \\
G\text{-spaces} \arrow[dr, hook] & & \{U_i\}_i \simeq X \arrow[dl, hook'] \\
 & \text{local quotient groupoids} \arrow[d, hook] & \\
 & \text{topological stacks} &
\end{tikzcd}
$$

The following material is a summary of the appendix of [7].

Definition 3.3.1. A *groupoid* $(X_0, X_1) = (X_1 \rightrightarrows X_0)$ is a groupoid object in the category Top of spaces.

Remark 3.3.2. Groupoids, (continuous) functors, and (continuous) natural transformations form a 2-category TGrpd.

Recall that the *nerve* of a groupoid is a simplicial space $X_\bullet$, where a point in X_n is an n-tuple of composable morphisms (points in X_1). The *geometric realization* $|X|$ of a groupoid X is the geometric realization of its nerve, i.e.

$$
|X| = \big(\coprod_n X_n \times \Delta^n\big)/\sim .
$$

Example 3.3.3. Spaces as groupoids.
Let X be a space. Then $(X, X) = (X \rightrightarrows X)$ is a groupoid, where all maps are the identity. Note that the associated map $\mathsf{Top} \to \mathsf{TGrpd}$ (with any appropriate meaning of Top) realizes Top as a full subcategory of TGrpd. In light of this, and the fact that $|(X, X)| \simeq X$, the groupoid (X, X) is also denoted X.

Example 3.3.4. Groups as groupoids.
Let G be a (topological) group. Then $(*, G) = (G \rightrightarrows *)$ is a groupoid where composition is defined by the group operation. Similar to Example 3.3.3, $\mathsf{TGrp} \to \mathsf{TGrpd}$ is a full subcategory. In light of the fact that $|(*, G)| \simeq BG$, the groupoid $(*, G)$ is sometimes denoted BG. This notation is not ideal, since the space BG contains less information than the groupoid BG (this difference is detected by the Atiyah-Segal completion theorem). In light of the next example, I prefer the notation $*/\!\!/G$ for this groupoid. (Alternatively, one can use BG to denote the groupoid, and $|BG|$ for the space.)

Example 3.3.5. G-spaces as groupoids.

Let X be a G-space. Then $X /\!\!/ G := (X, G \times X) = (G \times X \underset{t}{\overset{d}{\rightrightarrows}} X)$ is a groupoid, where $d(g, x) = x$, $t(g, x) = gx$, identities are given by the identity of G, and composition is defined by composition in G. In other words, a morphism from $x \in X$ to $y \in X$ is an element of G that sends x to y. The notation is motivated by the fact that $|X /\!\!/ G|$ is the homotopy quotient of X by G. This identification realizes the category RTop of G-spaces, that we previously used as the domain for equivariant K-theory, as a subcategory of TGrpd.

Example 3.3.6. Open covers as groupoids.
Let $\{U_i\}_i$ be an open cover of a space X. Then $\check{C}(\{U_i\}_i) := (\coprod_i U_i, \coprod_{i,j} U_i \cap U_j)$ is a groupoid with domain and target maps induced by the inclusions $U_i \cap U_j \hookrightarrow U_i$ and $U_i \cap U_j \hookrightarrow U_j$, and identity map given by the identity $U_i \xrightarrow{\mathrm{id}} U_i \cap U_i = U_i$. In other words, a morphism from $x_i \in U_i$ to $x_j \in U_j$ is an element $x_{ij} \in U_i \cap U_j$ such that $x_{ij} = x_i = x_j \in X$.

The examples above show that the category TGrpd contains the objects we want it to. We still need to show that an open cover of a space X is equivalent to X under an appropriate notion of equivalence.

Definition 3.3.7. A (continuous) functor of groupoids $F : X \to Y$ is a *local equivalence* if

- F is fully faithful,
- F is essentially surjective,
- F "admits local inverses".

The third condition is slightly more technical [7], but this is the main idea.

Remark 3.3.8. The equivalence relation generated by local equivalences is called *weak equivalence*. Given a groupoid X there is an associated *topological stack* $\tilde{X}$ defined as (the stackification of) $\tilde{X}(U) = \mathrm{Hom}_{\mathsf{TGrpd}}(U, X)$. Then two groupoids are weakly equivalent if and only if their associated topological stacks are equivalent. In light of this, going forward we will implicitly identify groupoids up to local equivalence with topological stacks.

Example 3.3.9. Let $\{U_i\}_i$ be an open cover of a space X. Then the projection

$$\begin{tikzcd} \coprod_{i,j} U_i \cap U_j \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & X \arrow[d, shift left] \arrow[d, shift right] \\ \coprod_i U_i \arrow[r] & X \end{tikzcd}$$

is a local equivalence $\check{C}(\{U_i\}_i) \to X$.

Example 3.3.10. Let $P \to X$ be a principal G-bundle. Then there is a local equivalence $P /\!\!/ G \to X$.

Before we move on, we need to restrict the category TGrpd to those groupoids that locally look like the groupoids in Example 3.3.5. In other words, we consider the subcategory that is just big enough to include the examples we are interested in. The reason this restriction is necessary is explained in [7].

Definition 3.3.11. A groupoid X is a *local quotient groupoid* if it is locally weakly equivalent to $S /\!\!/ G$ for a compact Lie group G acting on a Hausdorff space S. Here, *locally* means over an open cover of the object space X_0. Let $\mathsf{LQGrpd} \subset \mathsf{TGrpd}$ be the full subcategory of local quotient groupoids.

3.4 The 2-category of algebras and bimodules.

Now, we want to represent twists of K-theory as finite dimensional bundles over local quotient groupoids. We need to use 2-categorical structure in order to represent all twists by finite dimensional bundles. To this end, we introduce a 2-category that will serve as a classifying space for bundles over a local quotient groupoid. See [5] for more details on the material in this section.

Definition 3.4.1. Let Alg be the 2-category where

- objects are $\mathbb{Z}/2$-graded $\mathbb{C}$-algebras,
- a 1-morphism $A_0 \to A_1$ is a $\mathbb{Z}/2$-graded (A_1, A_0)-bimodule,
- 2-morphisms are homomorphisms of bimodules.

Then Alg is a symmetric monoidal 2-category under the operation $\otimes_{\mathbb{C}}$. Let $\mathsf{Alg}^\times \subset \mathsf{Alg}$ be the subcategory of invertible objects, invertible 1-morphisms, and invertible 2-morphisms. Further, we can enrich the hom-sets $\mathrm{Hom}_{\mathsf{Alg}^\times(A_0,A_1)}(M_0, M_1)$ of 2-morphisms with a topology to obtain a topologically enriched (Picard) 2-groupoid $\mathsf{cAlg}^\times$. Calculating the homotopy groups of the geometric realization, we find

$$\pi_n(\mathsf{cAlg}^\times) = \begin{cases} \mathbb{Z}, & n = 3 \\ 0, & \text{else} \\ \mathbb{Z}/2, & n = 1 \\ \mathbb{Z}/2, & n = 0. \end{cases}$$

These are the homotopy groups we need to classify the types of twists discussed after Proposition 3.1.1, as well as twists by cohomological degree. In other words, if X is a local quotient groupoid, then

$$\left\{\begin{array}{c}\text{continuous 2-functors}\\ X \to \mathsf{cAlg}^\times\end{array}\right\} \xleftrightarrow{\sim} H^0(X;\mathbb{Z}/2)\times H^1(X;\mathbb{Z}/2)\times H^3(X;\mathbb{Z}).$$

Remark 3.4.2. We have not yet said what it means to evaluate ordinary cohomology on a local quotient groupoid. Define $H^k(X;R) := H^k(|X|;R)$. In light of Example 3.3.5, this includes the ordinary Borel equivariant cohomology of a G-space as a special case. Thus, for equivariant K-theory, twists over a G-space X are classified by $H^0_G(X;\mathbb{Z}/2)\times H^1_G(X;\mathbb{Z}/2)\times H^3_G(X;\mathbb{Z})$.

Following the discussion above, we'll consider twists of K-theory to be maps $\tau : X \to \mathsf{cAlg}^\times$. Such a map can be though of as a bundle of invertible algebras $\tau = (A, B, \lambda)$, in the following way.

- On X_0, τ determines a fiber bundle of invertible (= central simple) $\mathbb{Z}/2$-graded algebras $A \to X_0$.
- On X_1, τ determines a $\mathbb{Z}/2$-graded vector bundle $B \to X_1$ which is an invertible (p_0^*A, p_1^*A)-bimodule.
- On X_2, τ determines an isomorphism λ of (bundles of) bimodules, which at a point $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2) \in X_2$, is an isomorphism $\lambda_{f_2,f_1} : B_{f_2} \otimes_{A_{x_1}} B_{f_1} \xrightarrow{\sim} B_{f_2f_1}$.
- On X_3, there is an associativity condition on the λ's relating the bimodules B over the various composites of a point $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} x_3) \in X_3$.

This realizes our goal of representing twists of K-theory by finite dimensional geometric objects.

Example 3.4.3. Let $X = \check{C}(\{U_i\}_i)$. We obtain a general class of twists by letting A be the trivial bundle $\mathbb{C}$. In this case,

- B assigns a line bundle $L_{ij} \to U_i \cap U_j$ to each i, j.
- λ specifies an isomorphism $L_{jk} \otimes L_{ij} \xrightarrow{\sim} L_{ik}$ over $U_i \cap U_j \cap U_k$.
- There's an associativity condition on the λ's over $U_i \cap U_j \cap U_k \cap U_l$.

There are the twists that are described in [7].

Example 3.4.4. Let $X = * /\!\!/ G$. Then

$$\left\{\begin{array}{c}\text{invertible algebra}\\ \text{bundles } (\mathbb{C}, B, \lambda)\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\mathbb{Z}/2\text{-graded central}\\ \text{extensions of } G \text{ by } \mathbb{C}^\times\end{array}\right\}.$$

In light of this example, Freed-Hopkins-Teleman [7] refer to more general twists over groupoids as "graded central extensions".

Example 3.4.5. Let G be a compact connected Lie group. Then LG acts on the path space PG, and G acts on itself, both by conjugation. There is a local equivalence of groupoids

$$PG /\!\!/ LG \xrightarrow{\sim} G /\!\!/ G.$$

Using this local equivalence, we can use graded central extensions $\tilde{LG} \to LG$ to define twists of $G/\!\!/G$. This is the groupoid of interest in (1). Here we find a relationship between K-theory twists over $G /\!\!/ G$ and the group LG as in the main theorem (1) of [7].

4 Twisted K-theory

Now that we've surveyed various ways of representing twists τ of K-theory, we move on to describe the τ-twisted K-theory of a local quotient groupoid.

4.1 Twisted vector bundles.

We start by introducing one way to define classes in twisted K-theory using the geometric model for twists discussed in section 3.4. The idea is based on the representation of (untwisted) K-theory classes by ($\mathbb{Z}/2$-graded) vector bundles.

Definition 4.1.1. Let $\tau = (A, B, \lambda)$ be a twist over a local quotient groupoid X. A *τ-twisted vector bundle* $E = (E_0, \psi)$ over X is

- an A-module (bundle) $E_0 \to X_0$,
- an isomorphism $\psi : B \underset{p_1^* A}{\otimes} p_1^* E_0 \xrightarrow{\sim} p_0^* E_0$ of $p_o^* A$-modules over X_1,
- satisfying a condition on X_2.

Remark 4.1.2. While τ-twisted vector bundles do represent classes in τ-twisted K-theory, we won't go too far in pursuing this model here, since they are not enough to represent *all* τ-twisted K-theory classes. Similar to the situation with Atiyah-Segal twists (Remark 3.2.1 and [4]), only torsion classes arise as (finite dimensional) twisted vector bundles. Harnessing the extra 2-categorical structure allowed us to recover all twists, but we still don't recover all twisted K-theory *classes.*

4.2 Twisted K-theory via Fredholm operators.

See [7] for more details on the material in this section. We want a complete model for twisted K-theory that is more geometric in nature than Definition 2.2.1. To this end, recall that ordinary K-theory is represented by a space of Fredholm operators on a Hilbert space $\mathcal{H}$:

$$\begin{aligned} K(X) &\cong [X, \mathrm{Fred}(\mathcal{H})] \\ &\cong \pi_0 \mathrm{map}(X, \mathrm{Fred}(\mathcal{H})) \\ &\cong \pi_0 \Gamma(X \times \mathrm{Fred}(\mathcal{H}) \to X). \end{aligned}$$

Using this point of view, we can replace the trivial bundle of Fredholm operators appearing above with a twisted one. Define *τ-twisted Hilbert bundle* as in Definition 4.1.1, replacing the words "vector space" with "Hilbert space".

Proposition 4.2.1. For any twist τ over a local quotient groupoid X, there exists a *locally universal* τ-twisted Hilbert bundle $\mathcal{H}$, i.e. one that all other τ-twisted Hilbert bundles locally embed into.

Given a locally universal τ-twisted Hilbert bundle $\mathcal{H}$, there is a bundle $\mathrm{Fred}^{(0)}(\mathcal{H}) \to X$ of odd skew-adjoint Fredholm operators. Further we can define $\mathrm{Fred}^{(n)}(\mathcal{H}) \subset \mathrm{Fred}^{(0)}(\mathbb{C}\mathrm{l}_n \otimes \mathcal{H})$ as the odd Fredholm operators that are also $\mathbb{C}\mathrm{l}_n$-linear, where $\mathbb{C}\mathrm{l}_n$ is the Clifford algebra associated to $\mathbb{C}^n$ with $v^2 = -\langle v, v \rangle$.

Definition 4.2.2. Let τ be a twist of K-theory over a local quotient groupoid X. Define

- $\underline{K}^\tau(X)_n := \Gamma(\mathrm{Fred}^{n \bmod 2}(\mathcal{H}) \to X)$, and $\underline{K}^\tau(X)$ for the resulting Ω-spectrum;
- $\underline{K}^\tau(X, A) :=$ homotopy fiber of $\underline{K}^\tau(X) \to \underline{K}^\tau(A)$;
- $K^{n+\tau}(X, A) := \pi_{-n}\underline{K}^\tau(X, A) \cong \pi_0 \underline{K}^\tau(X, A)_n$.

The groups $K^{n+\tau}(X, A)$ satisfy the typical properties of a cohomology theory:

- functoriality,
- homotopy invariance,
- there are long exact sequences,
- excision,
- $\coprod \mapsto \prod$,
- there is a product structure.

Example 4.2.3. Let $X = S /\!\!/ G$ be a G-space, and let τ be a twist over X corresponding to a central extension $U(1) \to G^\tau \to G$ with the trivial even grading. Then

$$K^{n+\tau}(X) \subset K^n_{G^\tau}(S)$$

is a direct summand (corresponding to $U(1)$ acting by its standard representation).

5 Computation of $K^{k+\tau}_{SU(2)}(SU(2))$

This computation is from [7].

Let $X = SU(2)/\!\!/SU(2)$, where $SU(2)$ acts on itself by conjugation. Then

- $H^1(X;\mathbb{Z}/2) = H^1_{SU(2)}(SU(2);\mathbb{Z}/2) = 0$, and
- $H^3(X;\mathbb{Z}) = H^3_{SU(2)}(SU(2);\mathbb{Z}) \cong \mathbb{Z}$,

so twists over X (other than the degree $= k$) correspond to $\tau = n \in \mathbb{Z}$.

Let $\{U_+, U_-\}$ be the open cover of $SU(2) \simeq S^3$ obtained by deleting -1 and $+1$, respectively. By the long exact sequence and excision properties, we have a Mayer-Vietoris sequence associated to this open cover.

- $U_\pm \underset{SU(2)}{\simeq} * \implies K^0_{SU(2)}(U_\pm) \cong R(SU(2)) \cong \mathbb{Z}[L, L^{-1}]^{\mathbb{Z}/2}$, where $\mathbb{Z}/2$ swaps L and L^{-1}.
- $U_+ \cap U_- \underset{SU(2)}{\simeq} SU(2)/U(1) \implies K^0_{SU(2)}(U_+ \cap U_-) \cong K^0_{U(1)}(*) \cong R(U(1)) \cong \mathbb{Z}[L, L^{-1}]$.

The Mayer-Vietoris sequence then manifests as

$$\begin{tikzcd}
 & & 0 \arrow[dll] \\
K^{0+\tau}_{SU(2)}(SU(2)) \arrow[r] & \mathbb{Z}[L,L^{-1}]^{\mathbb{Z}/2} \oplus \mathbb{Z}[L,L^{-1}]^{\mathbb{Z}/2} \arrow[r] & \mathbb{Z}[L,L^{-1}] \arrow[dll] \\
K^{1+\tau}_{SU(2)}(SU(2)) \arrow[r] & 0, &
\end{tikzcd}$$

from which it can be concluded that

$$K^{k+\tau}_{SU(2)}(SU(2)) \cong \begin{cases} 0, & k = 0 \\ \mathbb{Z}[L, L^{-1}]^{\mathbb{Z}/2}/(L^{n-1} + L^{n-3} + \cdots + L^{-(n-1)}), & k = 1 \end{cases}$$

This ring is isomorphic to the Grothendieck ring of positive energy representations of $LSU(2)$ at level $n-2$. This isomorphism is an instance of the Freed-Hopkins-Teleman [7] theorem, stated in equation (1) above.

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