Strong monoidal functors and modules

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I wrote this note because I wanted to know that a strong monoidal functor which is also an isomorphism of categories preserves (sub)categories of modules up to isomorphism (Proposition 11). A similar fact holds for strong monoidal equivalences (Proposition 13).

Definition 1. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be monoidal categories. A **lax monoidal functor** $(F, e, \mu_{x,y}) : (\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}) \to (\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ is

- a functor $F : \mathcal{C} \to \mathcal{D}$,
- a morphism $e : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$, and
- a natural transformation $\mu_{x,y} : F(x) \otimes_{\mathcal{D}} F(y) \to F(x \otimes_{\mathcal{C}} y)$,

such that

1. (**associativity**) for all $x, y, z \in \mathrm{obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
(F(x) \otimes_{\mathcal{D}} F(y)) \otimes_{\mathcal{D}} F(z) \arrow[r, "a_{\mathcal{D}}"] \arrow[d, "\mu \otimes \mathrm{id}"'] & F(x) \otimes_{\mathcal{D}} (F(y) \otimes_{\mathcal{D}} F(z)) \arrow[d, "\mathrm{id} \otimes \mu"] \\
F(x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{D}} F(z) \arrow[d, "\mu"'] & F(x) \otimes_{\mathcal{D}} F(y \otimes_{\mathcal{C}} z) \arrow[d, "\mu"] \\
F((x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{C}} z) \arrow[r, "F(a_{\mathcal{C}})"'] & F(x \otimes_{\mathcal{C}} (y \otimes_{\mathcal{C}} z))
\end{tikzcd}$$

   commutes, and

2. (**unitality**) for all $x \in \mathrm{obj}(\mathcal{C})$, the diagrams

$$\begin{tikzcd}
1_{\mathcal{D}} \otimes_{\mathcal{D}} F(x) \arrow[r, "e \otimes \mathrm{id}"] \arrow[d] & F(1_{\mathcal{C}}) \otimes_{\mathcal{D}} F(x) \arrow[d, "\mu"] \\
F(x) & F(1_{\mathcal{C}} \otimes_{\mathcal{C}} x) \arrow[l]
\end{tikzcd}$$

   and

$$\begin{tikzcd}
F(x) \otimes_{\mathcal{D}} 1_{\mathcal{D}} \arrow[r, "\mathrm{id} \otimes e"] \arrow[d] & F(x) \otimes_{\mathcal{D}} F(1_{\mathcal{C}}) \arrow[d, "\mu"] \\
F(x) & F(x \otimes_{\mathcal{C}} 1_{\mathcal{C}}) \arrow[l]
\end{tikzcd}$$

   commute.

Definition 2. A **strong monoidal functor** is a lax monoidal functor $(F, e, \mu_{x,y})$ such that e and $\mu_{x,y}$ are isomorphisms.

Definition 3. Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category. An **algebra object** (A, m, u) in $\mathcal{C}$ is

- an object $A \in \mathrm{obj}(\mathcal{C})$,
- a morphism $m : A \otimes A \to A$, and
- a morphism $u : 1 \to A$,

such that

1. (**associativity**) the diagram

$$\begin{tikzcd} (A \otimes A) \otimes A \arrow[r, "a"] \arrow[d, "m\otimes\mathrm{id}"'] & A \otimes (A \otimes A) \arrow[r, "\mathrm{id}\otimes m"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[rr, "m"'] & & A \end{tikzcd}$$

commutes, and

2. (**unitality**) the diagram

$$\begin{tikzcd} 1 \otimes A \arrow[r, "u\otimes\mathrm{id}"] \arrow[dr] & A \otimes A \arrow[d, "m"] & A \otimes 1 \arrow[l, "\mathrm{id}\otimes u"'] \arrow[dl] \\ & A & \end{tikzcd}$$

commutes.

Note that if A is an algebra object, and F is a lax monoidal functor, then the lax structure of F translates the multiplication of A into a multiplication on $F(A)$,

$$F(A) \otimes F(A) \xrightarrow{\mu} F(A \otimes A) \xrightarrow{F(m)} F(A).$$

Similarly, there is an induced unit

$$1_{\mathcal{D}} \xrightarrow{e} F(1_{\mathcal{C}}) \xrightarrow{F(u)} F(A).$$

Proposition 4. Let $(F, e, \mu_{x,y}) : (\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}) \to (\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be a lax monoidal functor. If (A, m, u) is an algebra object in $\mathcal{C}$, then $(F(A), F(m) \circ \mu_{A,A}, F(u) \circ e)$ is an algebra object in $\mathcal{D}$.

Proof. For associativity, we need to verify the commutativity of the diagram

$$\begin{tikzcd} (F(A) \otimes F(A)) \otimes F(A) \arrow[r, "a"] \arrow[d] & F(A) \otimes (F(A) \otimes F(A)) \arrow[r] & F(A) \otimes F(A) \arrow[d] \\ F(A) \otimes F(A) \arrow[rr] & & F(A) \end{tikzcd}$$

which expands to

$$\begin{tikzcd}
(F(A)\otimes F(A))\otimes F(A) \arrow[r, "a"] \arrow[d, "\mu\otimes\mathrm{id}"'] & F(A)\otimes(F(A)\otimes F(A)) \arrow[r, "\mathrm{id}\otimes\mu"] & F(A)\otimes F(A\otimes A) \arrow[r, "\mathrm{id}\otimes F(m)"] \arrow[d, "\mu"] & F(A)\otimes F(A) \arrow[d, "\mu"] \\
F(A\otimes A)\otimes F(A) \arrow[r, "\mu"] \arrow[d, "F(m)\otimes\mathrm{id}"'] & F((A\otimes A)\otimes A) \arrow[r, "F(a)"] \arrow[d, "F(m\otimes\mathrm{id})"] & F(A\otimes(A\otimes A)) \arrow[r, "F(\mathrm{id}\otimes m)"] & F(A\otimes A) \arrow[d, "F(m)"] \\
F(A)\otimes F(A) \arrow[r, "\mu"] & F(A\otimes A) \arrow[rr, "F(m)"] & & F(A).
\end{tikzcd}$$

This diagram is a concatenation of four rectangles. The top left rectangle commutes by associativity of the lax structure on F. The bottom right rectangle is F applied to the associativity diagram of A, so it commutes. The top right and bottom left squares commute by naturality of μ, so the entire diagram commutes.

For the unitality diagram,

$$\begin{tikzcd}
1_{\mathcal{D}}\otimes F(A) \arrow[r] \arrow[dr] & F(A)\otimes F(A) \arrow[d] & F(A)\otimes 1_{\mathcal{D}} \arrow[l] \arrow[dl] \\
& F(A) &
\end{tikzcd}$$

both sides behave the same, so we'll just consider the left triangle, which expands to

$$\begin{tikzcd}
1_{\mathcal{D}}\otimes F(A) \arrow[r, "e\otimes\mathrm{id}"] \arrow[d] & F(1_{\mathcal{C}})\otimes F(A) \arrow[r, "F(u)\otimes\mathrm{id}"] \arrow[d, "\mu"] & F(A)\otimes F(A) \arrow[d, "\mu"] \\
F(A) & F(1_{\mathcal{C}}\otimes A) \arrow[l] \arrow[r, "F(u\otimes\mathrm{id})"] & F(A\otimes A) \arrow[ll, bend left, "F(m)"]
\end{tikzcd}$$

which is a concatenation of two squares atop a (curved) triangle. The left square commutes by unitality of the lax structure of F. The right square commutes by naturality of μ. The bottom triangle is F applied to the unitality diagram of A, so it commutes. Thus, the entire diagram commutes.

□

Definition 5. Let (A, m, u) be an algebra object in a monoidal category $(\mathcal{C}, \otimes, 1)$. A **(left) module** (M, s) **over** A (or A-**module**) in $\mathcal{C}$ is

- an object $M \in \mathrm{obj}(\mathcal{C})$, and
- a morphism $s : A \otimes M \to M$,

such that

1. (**associativity**) the diagram

$$\begin{tikzcd}
(A\otimes A)\otimes M \arrow[r, "a"] \arrow[d, "m\otimes\mathrm{id}"'] & A\otimes(A\otimes M) \arrow[r, "\mathrm{id}\otimes s"] & A\otimes M \arrow[d, "s"] \\
A\otimes M \arrow[rr, "s"'] & & M
\end{tikzcd}$$

commutes, and

2. (**unitality**) the diagram

$$\begin{tikzcd}
1\otimes M \arrow[rr, "u\otimes\mathrm{id}"] \arrow[dr] & & A\otimes M \arrow[dl, "s"] \\
& M &
\end{tikzcd}$$

commutes.

As above, lax structures translate the action of an algebra A on a module M into an action of $F(A)$ on $F(M)$,

$$F(A)\otimes F(M) \xrightarrow{\mu} F(A\otimes M) \xrightarrow{F(s)} F(M).$$

Proposition 6. Let $(F, e, \mu_{x,y}) : (\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}) \to (\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be a lax monoidal functor, and let (A, m, u) be an algebra object in $\mathcal{C}$. If (M, s) is a module over A then $(F(M), F(s)\circ\mu_{A,M})$ is a module over $F(A)$.

Proof. The proof is similar to the proof of Proposition 4, but since I don't know where it's written, I'll write it. For associativity, we need to verify commutativity of the diagram

$$\begin{tikzcd}
(F(A)\otimes F(A))\otimes F(M) \arrow[r, "a"] \arrow[d] & F(A)\otimes(F(A)\otimes F(M)) \arrow[r] & F(A)\otimes F(M) \arrow[d] \\
F(A)\otimes F(M) \arrow[rr] & & F(M)
\end{tikzcd}$$

which expands to

$$\begin{tikzcd}
(F(A)\otimes F(A))\otimes F(M) \arrow[r, "a"] \arrow[d, "\mu\otimes\mathrm{id}"'] & F(A)\otimes(F(A)\otimes F(M)) \arrow[r, "\mathrm{id}\otimes\mu"] & F(A)\otimes F(A\otimes M) \arrow[r, "\mathrm{id}\otimes F(s)"] \arrow[d, "\mu"] & F(A)\otimes F(M) \arrow[d, "\mu"] \\
F(A\otimes A)\otimes F(M) \arrow[r, "\mu"] \arrow[d, "F(m)\otimes\mathrm{id}"'] & F((A\otimes A)\otimes M) \arrow[r, "F(a)"] \arrow[d, "F(m\otimes\mathrm{id})"] & F(A\otimes(A\otimes M)) \arrow[r, "F(\mathrm{id}\otimes s)"] & F(A\otimes M) \arrow[d, "F(s)"] \\
F(A)\otimes F(M) \arrow[r, "\mu"] & F(A\otimes M) \arrow[rr, "F(s)"] & & F(M).
\end{tikzcd}$$

This diagram is a concatenation of four rectangles. The top left rectangle commutes by associativity of the lax structure on F. The bottom right rectangle is F applied to the associativity diagram of M, so it commutes. The top right and bottom left squares commute by naturality of μ, so the entire diagram commutes.

For unitality, the relevant diagram is

$$\begin{tikzcd}
1_{\mathcal{D}} \otimes F(M) \arrow[rr] \arrow[dr] & & F(A) \otimes F(M) \arrow[dl] \\
& F(M) &
\end{tikzcd}$$

which expands to

$$\begin{tikzcd}
1_{\mathcal{D}} \otimes F(M) \arrow[r, "e\otimes\mathrm{id}"] \arrow[d] & F(1_{\mathcal{C}}) \otimes F(M) \arrow[r, "F(u)\otimes\mathrm{id}"] \arrow[d, "\mu"] & F(A) \otimes F(M) \arrow[d, "\mu"] \\
F(M) & F(1_{\mathcal{C}} \otimes M) \arrow[l] \arrow[r, "F(u\otimes\mathrm{id})"] & F(A \otimes M) \arrow[ll, bend left, "F(s)"]
\end{tikzcd}$$

which is a concatenation of two squares atop a (curved) triangle. The left square commutes by unitality of the lax structure of F. The right square commutes by naturality of μ. The bottom triangle is F applied to the unitality diagram of M, so it commutes. Thus, the entire diagram commutes.

□

We've established that lax monoidal functors take algebra objects to algebra objects and modules to modules. Next, we'd like to say something about the *categories* of modules over algebra objects.

Definition 7. Let (A, m, u) be an algebra object in a monoidal category $(\mathcal{C}, \otimes, 1)$, and let (M, s) and (N, t) be modules over A. A morphism $f : M \to N$ in $\mathcal{C}$ is called a **homomorphism** of A-modules if the diagram

$$\begin{tikzcd}
A \otimes M \arrow[r, "s"] \arrow[d, "\mathrm{id}\otimes f"'] & M \arrow[d, "f"] \\
A \otimes N \arrow[r, "t"] & N
\end{tikzcd}$$

commutes.

Since compositions of module homomorphisms are module homomorphisms, and identity morphisms of modules are module homomorphisms, A-modules and their homomorphisms form a subcategory of $\mathcal{C}$, denoted ${}_A\mathrm{Mod}$.

Proposition 8. If $f : M \to N$ is a homomorphism of A-modules and F is a lax monoidal functor, then $F(f)$ is a homomorphism of $F(A)$-modules.

Proof. We need to verify commutativity of the diagram

$$\begin{tikzcd}
F(A) \otimes F(M) \arrow[r] \arrow[d] & F(M) \arrow[d] \\
F(A) \otimes F(N) \arrow[r] & F(N)
\end{tikzcd}$$

which can be expanded to

$$\begin{tikzcd}
F(A) \otimes F(M) \arrow[r, "\mu"] \arrow[d, "\mathrm{id} \otimes F(f)"'] & F(A \otimes M) \arrow[r, "F(s)"] \arrow[d, "F(\mathrm{id} \otimes f)"'] & F(M) \arrow[d, "F(f)"] \\
F(A) \otimes F(N) \arrow[r, "\mu"] & F(A \otimes N) \arrow[r, "F(t)"] & F(N).
\end{tikzcd}$$

The left square commutes by naturality of μ, and the right square commutes since f is a homomorphism of A-modules and F is a functor. $\square$

Corollary 9. If $F : \mathcal{C} \to \mathcal{D}$ is a lax monoidal functor and $A \in \mathrm{obj}(\mathcal{C})$ is an algebra object, then $F({}_A\mathrm{Mod}) \subseteq {}_{F(A)}\mathrm{Mod}$.

We would like to know when this relationship is stronger than an inclusion. The situation is fairly simple for isomorphisms of categories.

Lemma 10. If $(F, e, \mu_{x,y})$ is a strong monoidal functor such that $F : \mathcal{C} \to \mathcal{D}$ is an isomorphism of categories, then $(F^{-1}, \tilde{e}, \tilde{\mu}_{x,y})$ is a strong monoidal functor, where $\tilde{e} = F^{-1}(e^{-1})$ and $\tilde{\mu}_{x,y} = F^{-1}(\mu^{-1}_{F^{-1}(x),F^{-1}(y)})$.

Proof. First, observe that $\tilde{e}$ and $\tilde{\mu}_{x,y}$ do indeed have the appropriate domains and codomains. The associativity and unitality diagrams can be obtained by inverting the arrows of the corresponding diagram for F and applying F^{-1} to the whole diagram. Thus, they commute. Naturality of $\tilde{\mu}$ follows from similar considerations. $\square$

Proposition 11. Let $(F, e, \mu_{x,y})$ be a strong monoidal functor such that $F : \mathcal{C} \to \mathcal{D}$ is an isomorphism of categories, and let $A \in \mathrm{obj}(\mathcal{C})$ be an algebra object. Then F restricts to an isomorphism of categories $F_A : {}_A\mathrm{Mod} \to {}_{F(A)}\mathrm{Mod}$.

Proof. Since F is an isomorphism, F restricts to an isomorphism ${}_A\mathrm{Mod} \to F({}_A\mathrm{Mod})$, so by Corollary 9, we just need to show that ${}_{F(A)}\mathrm{Mod} \subset F({}_A\mathrm{Mod})$. Since F^{-1} is also a lax monoidal functor by Lemma 10, we can apply Corollary 9 to F^{-1} and $F(A)$ to obtain the desired inclusion. $\square$

A similar fact holds for equivalences of categories, but it is more cumbersome.

Lemma 12. Let $(F, e, \mu_{x,y})$ be a strong monoidal functor such that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, and let $G : \mathcal{D} \to \mathcal{C}$ be an inverse equivalence with natural isomorphisms $\alpha : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\beta : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$. Then $(G^{-1}, \tilde{e}, \tilde{\mu}_{x,y})$ is a strong monoidal functor, where $\tilde{e} = G(e^{-1}) \circ \alpha_{1_{\mathcal{C}}}$ and $\tilde{\mu}_{x,y} = G\big((\beta_x \otimes \beta_y) \circ \mu^{-1}_{G(x),G(y)}\big) \circ \alpha_{G(x),G(y)}$.

Proof. Similar to Lemma 10, but with more commutative diagrams to unpack due to the extra natural isomorphisms. $\square$

Proposition 13. Let $(F, e, \mu_{x,y})$ be a strong monoidal functor such that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, and let $A \in \mathrm{obj}(\mathcal{C})$ be an algebra object. Then F restricts to an equivalence of categories $F_A : {}_A\mathrm{Mod} \to {}_{F(A)}\mathrm{Mod}$.

Proof. Since F is an equivalence, F restricts to an equivalence ${}_A\mathrm{Mod} \to F({}_A\mathrm{Mod}) \subseteq {}_{F(A)}\mathrm{Mod}$. Similarly, G restricts to an equivalence ${}_{F(A)}\mathrm{Mod} \to G({}_{F(A)}\mathrm{Mod}) \subseteq {}_{GF(A)}\mathrm{Mod} \cong {}_A\mathrm{Mod}$. Thus, G (together with the natural isomorphisms α, β from Lemma 12) provides an inverse equivalence for F_A. $\square$