ELLIPTIC COHOMOLOGY

— BY —

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<u>Goal</u>: Study homotopy theory ($\pi_* S$)

<u>Tool</u>: Cohomology theories (w/ alg. structure)

- H^*, K^* $\Rightarrow$ <u>Image(J)</u> (Garrett's talk)
  great! but limited
  ↳ geometric interpretations $\Rightarrow$ app'ns outside homotopy
  (e.g. diff. geo., Hodge th., Index thy, ...)

- MO^*, MSO^*, $MSpin^*$, MU^* $\rightarrow$ lots of information

→ Find other cohomology theories like H^*/K^*
- provides new/accessible info about $\pi_* S$
- geometric interpretation $\Rightarrow$ app'ns ("higher index thy")

Inspiration: Look for patterns among known cohom. thy's $\to$ they are often the home for various cobordism invariants.

Recall: (Thom isomorphism) $V \to X$ oriented v.b.

$$\Rightarrow \quad H^*(X) \longrightarrow H^{*+\operatorname{rank} V}(Th(V))$$
$$1 \longmapsto u_V$$

cob. inv. by Stokes's thm

$\downarrow$

fundamental class

$$u_{MSO} \in H^*(MSO) \iff MSO_* \longrightarrow H\mathbb{Z}_*$$

$$\& \quad MSO_*(X) \otimes_{MSO_*} H\mathbb{Z}_* \longrightarrow H_*(X) \quad \text{not surjective.}$$

Ex: (K-th) $V \to X$ complex v.b.

$\Rightarrow K^*(X) \cong K^*(Th(V))$

$\Rightarrow u_{MU} \in K^*(MU) \Longleftrightarrow MU_* \xrightarrow{\text{Todd genus}} K_*$

[Thm: (Conner-Floyd) $MU^*(X) \underset{\text{Todd}}{\otimes} K^*(pt) \cong K^*(X)$

Ex: $V \to X$ $Spin^c$ v.b. $\Rightarrow$ Thom iso

$\Rightarrow MSpin^c_* \xrightarrow{\hat{A}\text{-genus}} K_*$

[Thm: (Hopkins-Hovey) $MSpin^{c*}(X) \underset{\hat{A}}{\otimes} K^*(pt) \cong K^*(X)$

Similarly: $MSpin^*(X) \underset{\hat{A}}{\otimes} KO^*(pt) \cong KO^*(X)$

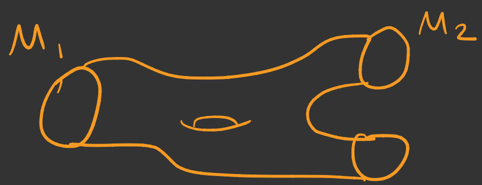
[<u>Moral</u>: Cobordism invariants (that are ring maps) often define cohomology theories.

<u>Def</u>: A <u>genus</u> is a ring homomorphism

$$\begin{tikzcd} MSpin_* \arrow[r] & MSpin^c_* \arrow[r] & MSO_* \arrow[r, "\Phi"] & R \\ & MU_* \arrow[u] & & \end{tikzcd}$$

oriented manifold $M \longmapsto \Phi(M) \in R$ s.t.

- $\Phi(M_1 \sqcup M_2) = \Phi(M_1) + \Phi(M_2)$
- $\Phi(M_1 \times M_2) = \Phi(M_1) \cdot \Phi(M_2)$ $*$
- $M_1 \underset{\text{cob.}}{\sim} M_2 \implies \Phi(M_1) = \Phi(M_2)$



<u>$\Phi = \hat{A}$: "Leray-Hirch"</u>

$(*)$ $\Phi(E) = \Phi(M_1) \cdot \Phi(M_2)$

$\forall$ bundles $M_1 \to E \to M_2$ w/ cpt conn'd str. gp & M_1 spin.

<u>Non-example</u>: Euler characteristic $\chi(M)$ is additive & multiplicative but $\chi(S^2) = 2$ & $\chi(\emptyset) = 0$.

$S^2 = \partial D^3$

Consider $MSO_* \otimes \mathbb{C} \xrightarrow{\Phi} \mathbb{C}$

$$\text{(Thom)}\ \cong$$

$$\mathbb{C}[\mathbb{CP}^2, \mathbb{CP}^4, \dots]$$

$$\int_0^x \left(\sum_{n=0}^{\infty} \Phi(\mathbb{CP}^{2n}) t^{2n} \right) dt$$

$$\parallel$$

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} \Phi(\mathbb{CP}^{2n}) x^{2n+1}$$

$$\parallel \colon$$

$$\log_\Phi(x)$$

Def: A genus $\Phi : MSO_* \to \mathbb{C}$ is <u>elliptic</u> if $\log_\Phi(x)$ is an elliptic integral, i.e.

$$\log_\Phi(x) = \int_0^x (1 - 2\delta t^2 + \varepsilon t^4)^{-\frac{1}{2}} dt,$$

for some $\delta, \varepsilon \in \mathbb{C}$

go back to $\hat{A}$ property w/ bundles of spin manifolds.

Ex: $(\delta, \varepsilon) = (-\frac{1}{8}, 0)$ $\displaystyle\int_0^x \frac{1}{\sqrt{1+(\frac{1}{2}t)^2}}\,dt = 2\sinh^{-1}(\tfrac{1}{2}x) = \log_{\hat{A}}(x)$

Thm: (Ochanine, Taubes) $\Phi : MSO_* \longrightarrow \mathbb{C}$ is elliptic $\iff$ Φ satisfies $(*)$

↳ elliptic genera "look like" genera that define cohom.

Q: Given an elliptic genus $\Phi : MSO_* \longrightarrow \mathbb{C}$, is $X \longmapsto MSO^*(X) \otimes_{\Phi} \mathbb{C}$ a cohomology theory.

A: no.

$\underline{\text{Q}}$: when do ring maps $MG_* \xrightarrow{\Phi} R$ induce cohom. thy's : $X \longmapsto MG^*(X) \underset{\Phi}{\otimes} R$?

$\underline{\text{Thm}}$: (Landweber) If $MU_* \xrightarrow{\Phi} R$ is "Landweber exact" then $MU^*(X) \underset{\Phi}{\otimes} R$ is a cohomology theory.

$\underline{\text{Thm}}$: (Quillen)

$$\left(\begin{array}{c}\text{ring maps}\\ MU_* \xrightarrow{\Phi} R\end{array}\right) \longleftrightarrow \left(\begin{array}{c}\text{formal group laws}\\ \text{over } R\end{array}\right)$$

For $\Phi : MSO_* \to R \implies F_\Phi(x,y) = \log_\Phi^{-1}(\log_\Phi(x) + \log_\Phi(y))$

elliptic $\implies \qquad = \dfrac{x\,r(x) + y\,r(x)}{1 - \varepsilon x^2 y^2}$

Consider the "universal" elliptic genus, where δ, ε are indeterminate, i.e. $\varphi : MSO_* \to \mathbb{C}[\delta, \varepsilon]$

Thm: $\varphi : MSO_* \longrightarrow \mathbb{Z}[\tfrac{1}{2}][\delta, \varepsilon]$

Thm: (LRS) The composite $MU_* \longrightarrow MSO_* \xrightarrow{\varphi} \mathbb{Z}[\tfrac{1}{2}][\delta, \varepsilon, \Delta^{-1}]$ is Landweber exact.

($\Delta = \varepsilon^2(\delta^2 - \varepsilon) =$ discriminant)

$$\Rightarrow \quad Ell^*(X) = MU^*(X) \otimes_{\varphi} \mathbb{Z}[\tfrac{1}{2}][\delta, \varepsilon, \Delta^{-1}]$$

$$\left(= MSO^*(X) \otimes_{\varphi} \mathbb{Z}[\tfrac{1}{2}][\delta, \varepsilon, \Delta^{-1}] \right)$$

is a cohom. theory called **Elliptic cohomology**

What is elliptic cohomology geometrically?

Recall a characteristic class is a natural transformation

$$c : \mathrm{Vect} \longrightarrow H^*$$

$\rightsquigarrow$ often descend to $\quad : KO \longrightarrow H^*$

Given $\circlearrowleft$ $\Phi_c(M) := \langle c(TM), [M] \rangle$

is a cobordism invariant by Stokes's thm.

Thm: $\forall$ genus $\Phi : MSO_* \longrightarrow \mathbb{C}$ $\exists$ char. class

$H_\Phi : KO \longrightarrow H^{ev}(\ ; \mathbb{C})$ s.t.

- (stable) $H_\Phi(1) = e \neq 0 \in H^*(pt; \mathbb{C}) \cong \mathbb{C}$
- (exponential) $H_\Phi(E \oplus F) = H_\Phi(E) \cdot H_\Phi(F)$
- $\Phi(M) = \langle H_\Phi(TM), [M] \rangle$.

Recall: $ch : K(X) \otimes \mathbb{C} \xrightarrow{\sim} H^{ev}(X; \mathbb{C})$

$\Longrightarrow$ can restate thm in terms of K-theory.

Thom isom's $\Longrightarrow$ Gysin ("pushforward"/integration) maps

$M \longrightarrow pt \quad \rightsquigarrow \quad \pi_!^M : E^*(M) \longrightarrow E^*(pt)$.

"Diff. Riemann-Roch" Thm (A-H):

$$
\begin{tikzcd}
MSO_* \arrow[r, "T"] \arrow[ddrr, "a_K \pi_!^M K_\Phi T =: \Phi_K"'] \arrow[ddrrr, "\Phi = a_H \pi_!^M H_\Phi T" near start] & KO \arrow[r, dashed, "K_\Phi"] \arrow[rr, bend left, "H_\Phi"] & K^*(M) \otimes \mathbb{C} \arrow[r, "\hat{A}(M)\cdot ch", "\sim"'] \arrow[d, "\pi_!^M"] \arrow[dr, phantom, "\circlearrowleft \quad \cancel{\circlearrowleft}"] & H^*(M;\mathbb{C}) \arrow[d, "\pi_!^M = \langle -, [M] \rangle"] \\
 & & K^*(pt) \otimes \mathbb{C} \arrow[r, "ch", "\sim"'] \arrow[d, "a_K", "\cong"'] & H^*(pt;\mathbb{C}) \arrow[d, "\cong", "a_H"'] \\
 & & \mathbb{C} \arrow[r, "ch"] & \mathbb{C}
\end{tikzcd}
$$

$$\Rightarrow \quad ch\, \Phi_K = \hat{A}^{-1}\, \Phi$$

Why does this help?

① Atiyah-Singer $\Rightarrow$ K-th: M spin $\Rightarrow$

Dirac operator

$$\pi_!^M(E) = \mathrm{Index}(\not{D}) \in \mathbb{Z} \ !$$

Similarly $\rightsquigarrow$ universal elliptic genus is integral on spin manifolds

② spin

$$G \circlearrowright M \Rightarrow K_G^*(M) =: K^*(M/\!/G)$$

$$\& \quad \Phi_K(M/\!/G) \in K^*(pt/\!/G) \otimes \mathbb{C} \cong R(G) \otimes \mathbb{C}$$

Thm: A genus $\Phi : MSO_* \to \mathbb{C}$ is elliptic iff $\forall$ cpt conn'd $G \circlearrowright M$ spin: $\Phi_K(M/\!/G) = \Phi_K(M)$

↳ Due to the theory of maximal tori, enough to check this for $S^1 \circlearrowright M$.

Witten: the universal elliptic genus arises as the S^1-equivariant index of a "Dirac" operator on the free loop space $\mathcal{L}M \circlearrowleft S^1$ by rotation.

Somehow this intuition comes from QFT...

$$\mathrm{Bord}_d \longrightarrow \mathrm{Vect}$$

<u>$d=1$:</u> vector bundles with connection

$\Rightarrow$ K-theory.

<u>$d=2$:</u> $Ell^*(X)$?