Math 199, Spring 2022
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2/23/22

Participation Assignment 6 - Applications of integrals: (curved) length and area

Estimated Time: Less than 1 hour.

Goals: Better understand how integrals are used to compute quantities like (curved) length and area.

1) Summarize what you remember from our "general strategy for setting up a definite integral", and briefly explain how you would apply this to find arc length and surface area.

2) Recall that if we want to apply our general strategy to find the area under the graph of $f(x) = x$ on $[0, 1]$, using x as our "conveniently chosen variable", we can use rectangles as our "local estimates" for the area (think Riemann sums). Draw a sketch of this situation, set up and compute the corresponding definite integral, and verify that this answer is consistent with the area of the shape you know the region to be.

3) Let's try to apply a similar idea to compute the *length* of the graph of $f(x) = x$ on $[0, 1]$. This time, instead of using rectangles to approximate the area underneath the graph, just use the lengths of the tops of the rectangles to approximate the length of the graph over each small interval. Draw a sketch of this situation, and set up and compute the corresponding definite integral.

4) Compute the length of the graph of $f(x) = x$ without calculus. (*Hint: the Pythagorean theorem.*)

5) If you did them correctly, your answers to problems (3) and (4) should be different. Why didn't the method we used in problem (3) get us the correct answer?

6) Based on what you've seen above, what needs to be true of our "local estimate" in general, if we want the corresponding definite integral to give us the correct value?

7) Using your knowledge of the "differential" ds, redo problem (3) correctly.

8) As you might've noticed by this point in your math education, it is often useful to think about mathematical concepts as *functions*. Given a function $f(x)$ which is differentiable on the whole real line, define a new function

$$s_f(x) = \text{the arc length of the graph of } f \text{ on the interval } [0, x].$$

Find $s_f(x)$ for the function $f(x) = x$.

9) There is a differential "dA" that measures "local area" of a surface so that

$$\text{Area} = \int_I dA,$$

In general, multivariable notions are required to make sense of this for an arbitrary surface. It turns out that we know how to find dA for surfaces of revolution, since they only depend upon a single variable.

(a) Write down a formula for dA for the surface obtained by revolving the graph of $f(x)$ about the x-axis. Explain why this provides a local estimate of the surface area using the language of our general strategy.

(b) Use your reasoning above to write down a formula for dA for the surface obtained by revolving the graph of $f(x)$ about the line $y = b$, where $f(x) > b$ for all x (i.e. the graph lies above the axis of revolution).

10) Find the area of the surface obtained by revolving the graph of $f(x) = 3x^3 - x - 1$ on the interval $x \in [1, 2]$ about the line $y = -1$. (Start by drawing a sketch.)