Math 199, Fall 2022
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Participation assignment 4 - Partial fractions

Estimated time: 1 hour (for problems 1-3).

Point value: 3 points.

Goals: The main goal for this worksheet is to understand how to apply the method of partial fractions. In other words, we want to write a rational function as a sum of simpler rational functions that are easier to integrate. While our ultimate goal is integration, the method itself is purely algebraic, and this worksheet focuses on the algebraic aspects.

1) Recall the three cases that the denominator of a rational function may fall under:

Case I: the denominator factors into *distinct* linear terms:

$$f(x) = \frac{P(x)}{C(x-a_1)\cdots(x-a_n)}, \quad \text{where no two of the } a_k\text{'s are the same.}$$

Case II: the denominator factors into linear terms which are *not all distinct*:

$$f(x) = \frac{P(x)}{C(x-a_1)^{b_1}\cdots(x-a_n)^{b_n}}.$$

Case III: the denominator factors into linear and (irreducible) quadratic terms: e.g.

$$f(x) = \frac{P(x)}{C(x-a_1)\cdots(x-a_n)(x^2+b_1x+c_1)\cdots(x^2+b_mx+c_m)}.$$

First, when is a quadratic polynomial irreducible?

Next, write down a general form for the partial fraction decomposition for each of the three cases above.

Case I:

Case II:

Case III:

2) For each of the following rational functions:

- determine the degrees of the numerator and the denominator polynomials;
- factor the denominator, and state which case from problem (1) it falls under;
- find the partial fraction decomposition.

(a) $\dfrac{1}{x^2 + 7x + 10}$

(b) $\dfrac{4}{x^2 - 9}$

(c) $\dfrac{x}{x^2 + 1}$

(d) $\dfrac{1}{x^2 - 6x + 9}$

(e) $\dfrac{-2x}{x^3 - x^2 + 2x - 2}$

(f) $\dfrac{x^2 + 3}{(x + 2)(x^2 - x - 6)}$

3) You may notice that your decomposed answers to problem (2) look like they shouldn't be too hard to integrate. This doesn't always work out. Particularly, when the degree of the numerator is not smaller than the degree of the denominator. In this problem, we'll handle this situation.

Let

$$f(x) = \frac{2x^2 - 9x - 2}{x - 5}.$$

We want to divide the polynomial in the numerator by the polynomial in the denominator using a process called "long division", which is similar to the long division of whole numbers that you know. The goal is to write

$$2x^2 - 9x - 2 = Q(x)(x - 5) + R,$$

where $Q(x)$ is a *linear* (= degree 1) polynomial, and R is a number (= degree 0 polynomial).

Write $Q(x) = ax + b$, and use the above equation to solve for a, b, and R.

Use your new identity to rewrite $f(x)$ as a sum of two (or three) simple rational functions.

Compute $\int f(x)dx$.

4) *Bonus:* If you still have time before the break, integrate the rational functions from problem (2).

(a)

(b)

(c)

(d)

(e)

(f)