Math 199, Fall 2023
Yigal Kamel
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Participation assignment 11 - The mean value theorem

Estimated time: 30-50 minutes.

Point value: 3 points.

Goals: Understand what the mean value theorem (MVT) says, and learn some ways it can be applied to solve problems.

1) Before we get to *using* the mean value theorem, let's look at some examples to help us understand what it says a bit better. For each of the following functions, find a specific x-value $x = c$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Include a sketch of the graph of f (practice curve sketching!), along with the "average" secant line from $(a, f(a))$ to $(b, f(b))$, and the tangent line to f at $x = c$.

(a) $f(x) = x^2 + 2x - 1$, $[a, b] = [0, 1]$.

(b) $f(x) = x + \frac{1}{x}$, $[a, b] = [\frac{1}{2}, 2]$.

(c) $\sin x + x$, $[a.b] = [0, 2\pi]$.

2) Sometimes, the mean value theorem can be confused with the intermediate value theorem. This problem is an exploration of certain cases where the conclusion of the MVT can actually be obtained from the IVT. Let $f(x)$ be a twice differentiable function whose 2nd derivative is continuous on $[a, b]$. Suppose $f'(x)$ and $f''(x)$ are never zero on $[a, b]$. Draw a few sketches of these types of situations, and give a brief informal argument that the MVT follows from the IVT in these scenarios.

Bonus: (return to this at the end if you like.) Make your argument mathematically rigorous.

Sometimes the MVT doesn't follow from the IVT.

3) Suppose you are studying a differentiable function which you only have partial knowledge of. In particular, you were able to figure out the following information:

$$f(x) = \begin{cases} x^2, & x \geq 1 \\ -x^2 & x \leq -1, \end{cases}$$

but you don't know the other values of $f(x)$.

Calculate the slope of $f(x)$ in the regions where you can.

What is the smallest value of the slope in these regions?

Show (using MVT) that there must be a point outside this region with a smaller slope, and identify the smallest slope that is necessarily attained.

Contrast this conclusion with what the intermediate value theorem could've told you.

Can an even smaller slope occur? Give an example or explain why not.