Morava's orbit picture & Morava stabilizer groups

References:

- Ravenel, "Nilpotence & periodicity in stable homotopy theory"
- Henn, "A mini-course on Morava stabilizer groups & their cohomology"
- Ravenel, "Complex cobordism & stable homotopy groups of spheres.

<u>Recall:</u> A power series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in R[\![x]\!]$ is a <u>strict isomorphism</u> if $a_0 = 0$ & $a_1 = 1$.

$\Gamma_R :=$ group of strict isomorphisms over R under functional composition.

$\Gamma_R \circlearrowright FGL(R)$ via $\gamma^{-1}(F(\gamma(x), \gamma(y)))$

$\Gamma := \Gamma_{\mathbb{Z}} \Rightarrow \Gamma \circlearrowright L$ also:

$G(x,y) =$ universal FGL over L

$$L = (\mathbb{Z}[x_1, x_2, \ldots])$$

$\gamma \in \Gamma \Rightarrow \gamma^{-1}(G(\gamma(x), \gamma(y)))$ FGL/L

$\Rightarrow$ classified by $\varphi_\gamma : L \longrightarrow L$

with inverse $\varphi_{\gamma^{-1}}$

$\Rightarrow \Gamma \circlearrowright L$ via φ_γ.

$\quad\quad\quad\quad \cong$

$\quad\quad\quad\quad MU_*(pt)$

<u>Turns out:</u> also $\Gamma \circlearrowright MU_*(X)$ $\circlearrowleft$

compatibly w/ $\Gamma \circlearrowright MU_*(pt)$

i.e. $x \in MU_*(X)$, $\gamma \in \Gamma$, $\lambda \in MU_*(pt)$:

$\gamma(\lambda x) = \gamma(\lambda)\gamma(x)$

homotopy category of finite CW complexes $= \mathcal{FH}$

category of fin. pres. graded L-modules w/ Γ-action compatible w/ $\Gamma \circlearrowright L$ $= \mathcal{C}\Gamma$

$$
\begin{tikzcd}
\mathcal{FH} \arrow[d, "\widetilde{MU}_*"'] & \supseteq & \mathcal{FH}_{(p)} \arrow[d, leftrightarrow, "\text{p-local}"] & = & \mathcal{F}_{p,0} & \supseteq \cdots \supseteq & \mathcal{F}_{p,n} \arrow[d, leftrightarrow, "\text{thick subcategories}"] & \supseteq \cdots \\
\mathcal{C}\Gamma & \supseteq & \mathcal{C}\Gamma_{(p)} & = & \mathcal{C}_{p,0} & \supseteq \cdots \supseteq & \mathcal{C}_{p,n} & \supseteq \cdots
\end{tikzcd}
$$

$\mathcal{F}_{p,n}$ = p-local finite CW complexes X s.t. $v_{n-1}^{-1}\widetilde{MU}_*(X) = 0$

$\mathcal{C}_{p,n}$ = p-local modules M s.t. $v_{n-1}^{-1}M = 0$.

<u>Morava's orbit picture</u> (early 70's)

analogous to above, but for classifying formal group laws.

<u>Recall</u>: $\mathrm{Hom}_{\mathrm{Rings}}(L, R) \cong FGL(R)$

$$\circlearrowright$$
$$\Gamma_R$$

<u>Ex</u>: $\Gamma \circlearrowright FGL(\mathbb{Z})$

Prop: Consider $\Gamma \circlearrowright FGL(\mathbb{Z})$.

(1) $F, G \in FGL(\mathbb{Z})$ are in the same Γ-orbit iff they are isomorphic $/\mathbb{Z}$.

(2) $F \in FGL(\mathbb{Z})$, the subgroup of Γ fixing F is the strict automorphism group of F.

(3) The strict automorphism groups of iso'c FGLs are conjugate in Γ.

Classifying FGLs over $\mathbb{Z}$ is hard.

easier over $k = \overline{\mathbb{F}}_p$.

$\Rightarrow \Gamma_k \circlearrowright FGL(k)$

$\Rightarrow \Gamma_k$-orbits $\leftrightarrow$ height of FGL.

<u>Prop:</u> Let $\begin{array}{ccc} FGL(k) & \cong & \mathrm{Hom}(L, k) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ F & \longmapsto & \theta \end{array}$. Then

F has height n iff $\theta(v_i) = 0 \ \forall i < n$

& $\theta(v_n) \neq 0$.

Y_n = height n T_k-orbit $\subseteq FGL(k)$.

$X_n = \bigcup_{i \geq n} Y_i \Rightarrow FGL(k) = X_1 \supseteq X_2 \supseteq \cdots \supseteq X_\infty$

↳ analog of thick subcat. thm for FGLs.

A representative $F_n \in Y_n$ is given by the FGL classified by the map $\mathbb{Z}_{(p)}[v_1, v_2, \ldots] \to \mathbb{F}_p$
$v_n \mapsto 1$ & other gen's $\mapsto 0$. Often called the
Honda FGL = FGL of $K(n)$ ← Morava K-theory

Morava Stabilizer groups

$$\mathbb{S}_n^s = \mathrm{Aut}_{strict}(F_n) = \mathrm{Stab}(F_n) \subseteq \Gamma_k \quad \text{(Ravenel)}$$

$$\cap \qquad = \text{(strict) Morava stabilizer group}$$

[full group of iso's of FGLs]

$$\mathbb{S}_n^f = \mathrm{Aut}_{full}(F_n) = \widetilde{\mathrm{Stab}}(F_n) \subseteq \tilde{\Gamma}_k \quad \text{(others)}$$

$$\cap \qquad = \text{(small) Morava stabilizer group}$$

$$\mathbb{G}_n = \mathbb{S}_n^f \rtimes \mathrm{Gal}(\mathbb{F}_{p^n} / \mathbb{F}_p)$$

[why? (D-H) $L_{K(n)} S^0 \simeq E_n^{h\mathbb{G}_n}$]

$$= \text{big Morava stabilizer group}$$

A description of S_n:

Recall $\mathbb{F}_p \rightsquigarrow \mathbb{Z}_p \cong W(\mathbb{F}_p)$

generalizes to $\mathbb{F}_{p^n} \rightsquigarrow W(\mathbb{F}_{p^n})$ with vectors

<u>Note:</u> $W(\mathbb{F}_{p^n})$ is a degree n extension of $\mathbb{Z}_p$.

- Frobenius lifts to $\sigma \circlearrowright W(\mathbb{F}_{p^n})$ with $\sigma(x) = x^p \pmod p$

adjoin a non-commuting variable

- $$E_n := W(\mathbb{F}_{p^n})\langle S \rangle \Big/ \begin{pmatrix} S^n = p \\ Sx = \sigma(x) S \end{pmatrix} \quad (= \mathcal{O}_n)$$

$$= \left\{ \sum_{i=0}^{n-1} a_i S^i \;\middle|\; a_i \in W(\mathbb{F}_{p^n}) \right\}$$

$$= \left\{ \sum_{i=0}^{\infty} e_i S^i \;\middle|\; e_i \in W(\mathbb{F}_{p^n}),\ e_i^{p^n} = e_i \right\}$$

$$\Rightarrow \quad E_n^{\times} = \left\{ \sum_{i=0}^{\infty} e_i S^i \in E_n \;\middle|\; e_0 \neq 0 \right\} \cong \mathbb{S}_n^f$$

<u>Prop:</u> $\mathbb{S}_n^s \cong \left\{ \sum_{i=0}^{\infty} e_i S^i \in E_n \;\middle|\; e_0 = 1 \right\} \subseteq E_n^{\times}$

Relation to the Hopf algebra of $K(n)$

$$K(n)_*(K(n)) \cong \Sigma(n) \otimes \Lambda(\tau_0, \tau_1, \ldots, \tau_{n-1})$$

where $\Sigma(n) = K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_*$

$=$ n^{th} <u>Morava Stabilizer algebra</u>

$$\cong K(n)_*[t_1, t_2, \ldots] \Big/ \left(t_i^{p^n} - v_n^{p^i - 1} t_i\right)$$

$$\mathbb{S}_n = \left\{ \sum_{i=0}^{\infty} e_i S^i \;\middle|\; e_i \in W(\mathbb{F}_{p^n}),\ e_i^{p^n} = e_i \right\}$$

$\Rightarrow$ each e_i is a continuous fcn on $\mathbb{S}_n$.

Let $S(n)$ = ring of such cts fcns

$$\cong \mathbb{F}_{p^n}[e_1, e_2, \ldots] \Big/ (e_i^{p^n} - e_i)$$

$$\cong \Sigma(n) \otimes_{K(n)_*} \mathbb{F}_{p^n}.$$

$\mathbb{S}_n$ as cohomology operators

Recall: Γ ~ multiplicative operators of MU

$\rightarrow \mathbb{S}_n$ ~ - - - - - - $K(n)$

$$FK(n)_*(X) = K(n)_*(X) \otimes_{K(n)_*} \mathbb{F}_{p^n}$$

$\Rightarrow \mathbb{S}_n \cong$ group of multiplicative operators of $FK(n)$.

Difference: $MU_*(MU)$ cannot be recovered from Γ, but $K(n)_*(K(n))$ can be recovered from $\mathbb{S}_n$.

Relevance for the ANSS:

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}BP_*/I_n) \cong \mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*)$$

↳ do computations in terms of the simpler Morava K-theories.

Group cohomology

$\mathbb{S}_n$ is a profinite group $\Rightarrow$ continuous (mod p) cohomology

Thm:

- $H^*(\mathbb{S}_n)$ finitely gen'd alg.
- $(p-1) \nmid n$: $H^i(\mathbb{S}_n) = \begin{cases} 0, & i > n^2 \\ H^{n^2-i}(\mathbb{S}_n), & 0 \le i \le n^2 \end{cases}$

  (Poincaré duality)
- $(p-1) \mid n$: $\exists\, x \in H^{2i}(\mathbb{S}_n)$ s.t.

  $H^*(\mathbb{S}_n)$ free $\mathbb{Z}/(p)[x]$ - module
- $U \overset{\text{open}}{\le} \mathbb{S}_n \Rightarrow H^*(U) \cong H^*(\mathbb{Z}_p^{n^2}) \cong \Lambda^{\bullet}(n^2)$

Thm: • All finite abelian subgroups of S_n are cyclic.

• S_n has an element of order p^{i+1} $\iff$ $(p-1)p^i \mid n$.