K(1)-local homotopy theory

References:

- Barthel-Beaudry, "Chromatic structures in stable homotopy theory."
- Ravenel, "Localization with respect to certain periodic homology theories".
- Henn, "A mini-course on Morava Stabilizer groups and their cohomology."

Topology → classify spaces up to homeomorphism

TOO HARD!

Homotopy theory → " " homotopy equivalence

kind of still too hard

↓

CW complexes

(<u>OR</u>: weak equivalences)

$\mathrm{Top} \supseteq W$ = subcategory of weak homotopy equivalences

$$\downarrow$$

$\mathrm{Top}[W^{-1}]$ (model cats / ∞-cats)

Similarly, $\mathrm{Sp} = \mathrm{Spectra}[W^{-1}]$

↳ still hard for now → homology

E = spectrum / homology theory E-equivalences

$$W_E = \{ X \xrightarrow{f} Y \mid E_* f : E_* X \xrightarrow{\sim} E_* Y \text{ iso} \}$$

(Sullivan, Adams) Bousfield

Let E be a spectrum.

$\exists!\ L_E : Sp \longrightarrow Sp$ with $\eta : \mathrm{id}_{Sp} \Rightarrow L_E$

such that

- $(\eta_X : X \to L_E X) \in W_E$
- $$\begin{tikzcd} X \arrow[r] \arrow[d, "\forall\, W_E \ni f"'] & L_E X \\ Y \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

⚠ If E (or X) is not connective, then $L_E X$ is not related to X by "localizing the homotopy groups"

e.g. $\pi_{-2} L_{KU} S^0 \cong \mathbb{Q}/\mathbb{Z}$

($L_{K(1)} S^0$ also not connective)

⚠ $L_E Sp$ not cocomplete

L_E doesn't commute with limits.

The Bousfield Lattice

- $\langle E \rangle :=$ equiv class under $E \sim F \iff L_E = L_F$.

$$\iff \left(E_* X = 0 \iff F_* X = 0 \right).$$

- $\langle E \rangle \leq \langle F \rangle$ if $\left(F_* X = 0 \Rightarrow E_* X = 0 \right)$.

  $\to$ $\wedge$ & $\vee$ descend to $\langle - \rangle$.

  $\to$ lattice $DL \supseteq$ Boolean algebra BA

Ex:

- $\langle S^0 \rangle \geq \langle MU \rangle \geq \langle BP \rangle \geq \langle BP\langle n \rangle \rangle \geq \langle E(n) \rangle \geq \langle K(n) \rangle$
- m≠n: $\langle K(n) \rangle$ & $\langle K(m) \rangle$ not comparable & $\wedge = \langle 0 \rangle$

Recall: $\pi_* BP = \mathbb{Z}_{(p)}[v_1, v_2, \dots] \rightsquigarrow BP\langle n\rangle, k(n)$

- $\pi_* BP\langle n\rangle = BP_* / (v_{n+1}, \dots)$
- $\pi_* k\langle n\rangle = BP_* / (p, v_1, \dots, v_{n-1}, v_{n+1}, \dots)$
- $E(n) = v_n^{-1} BP\langle n\rangle$ "Morava E-theory"
- $K(n) = v_n^{-1} k(n)$ Morava K-theory

Thm: $\langle E(n)\rangle = \langle v_n^{-1} BP\rangle = \bigvee_{m=1}^{n} \langle K(m)\rangle$

Moral: $K(n)$'s are $\binom{\text{not}}{\text{all}}$ "atoms" of Sp.

$E(1)$-local homotopy theory

$$KU_{(p)} = \bigvee_{i=1}^{p-1} E(1) \Rightarrow L_1 := L_{E(1)} = L_{KU_{(p)}}$$

In fact, $\langle KU \rangle = \langle KO \rangle \Rightarrow L_1 = L_{KO_{(p)}}$

"height 1 homotopy theory is topological K-theory"

Thm: (smash product thm)

$$L_1 X \simeq X \wedge L_1 S^0.$$

Note: for $p=2$ proof uses $KO_{(2)}$ as a rep. of $\langle E(1) \rangle$ instead of $KU_{(2)}$

$\Rightarrow L_1 Sp$ complete & L_1 commutes w/ colimits.

Cor: E = homology theory (e.g. BP)

$$E \wedge L_1 X \simeq X \wedge L_1 E$$

Pf: apply S.P.T. to X on left & E on right. //

↳ let's us compute homology on $L_1 Sp$

Ex: If $BP_* X \otimes \mathbb{Q} = 0$, then

$$BP_* L_1 X \cong v_1^{-1} BP_* X$$

→ But still need to know $L_1 S^0$.

The $E(1)$-local sphere

S.P.T. $\Rightarrow$ $L_1 S^0 \wedge L_1 S^0 \simeq L_1 L_1 S^0 \simeq L_1 S^0$

$\rightsquigarrow$ $L_1 S^0$ ring spectrum

Thm: $p > 2$.

$$\pi_i L_1 S^0 = \begin{cases} \mathbb{Z}_{(p)}, & i = 0 \\ \mathbb{Q}/\mathbb{Z}_{(p)}, & i = -2 \\ \mathbb{Z}/p^{j(i)}, & i = sp^k(2p-2)-1,\ p \nmid s,\ i \neq -1 \\ 0 & \text{else} \end{cases}$$

$=$ image of $\pi_* S^0$ detected by $\mathrm{Ext}^1_{BP_*BP}(BP_*, BP_*)$ in ANSS.

$\underline{p = 2.}$

$$\pi_i L_1 S^0 = \begin{cases} \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 , & i = 0 \\ \mathbb{Q}/\mathbb{Z}_{(2)} , & i = -2 \\ \mathbb{Z}_{(2)}/2s , & i = 8s-1 , \quad s \neq 0 \\ \mathbb{Z}/2 , & i = 8s , \quad s \neq 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 , & i = 8s+1 \\ \mathbb{Z}/2 , & i = 8s+2 \\ \mathbb{Z}/8 , & i = 8s+3 \\ 0 , & \text{else} \end{cases}$$

$K(1)$ - local homotopy theory

$L_1 S^0$ & $L_{K(1)} S^0$ related by a pullback square

$$\begin{tikzcd} L_1 S^0 = L_{E(1)} S^0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{K(1)} S^0 \arrow[d] \\ H\mathbb{Q} = L_{E(0)} S^0 \arrow[r] & L_{E(0)} L_{K(1)} S^0 \end{tikzcd}$$

$\Rightarrow$ modulo rational homology, pick which one you want to work with.

Thm: $\underline{p>2}$.

$$\pi_i L_{K(1)} S^0 = \begin{cases} \mathbb{Z}_p, & i = 0, -1 \\ 0, & i = -2 \\ \pi_i L_1 S^0, & \text{else} \end{cases}$$

$\underline{p=2}$.

$$\pi_i L_{K(1)} S^0 = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/2, & i = 0 \\ \mathbb{Z}_2, & i = -1 \\ \mathbb{Z}/2^{j(s)}, & i = 8s-1, \ i \neq -1 \\ 0, & i = -2 \\ \pi_i L_1 S^0, & \text{else} \end{cases}$$

The Morava Stabilizer group

$$\mathbb{G}_1 = \mathbb{S}_1 = \mathbb{Z}_p^\times$$

$\sim$ Adams operations in K-theory

- Even though Ravenel did it already, can also use (D-H) $L_{K(1)} S^0 \xrightarrow{\sim} K^{h\mathbb{Z}_p^\times}$ to compute $\pi_* L_{K(1)} S^0$ via K-A.S.S.

- Can also use a <u>finite resolution</u>.

<u>Idea</u>: decompose $L_{K(1)}S^0 \simeq K^{h\mathbb{Z}_p^\times}$ more.

Overview of the strategy

① exact sequence of $\mathbb{Z}_p^\times$-modules resolving $\mathbb{Z}_p$ $\rightsquigarrow$ apply $\mathrm{Hom}(-, K_*)$

$\Rightarrow$ s.e.s. of "Morava modules".

② Realize above as a fiber sequence of $K(1)$-local spectra.

↳ May not exist in general, use understanding of $K(n)$ to construct.

Example

$\mathbb{Z}_p^\times \cong \mu \times \mathbb{Z}_p$ where

$\underline{p>2}$: $\mu = C_{p-1}$, $\mathbb{Z}_p = \{a \in \mathbb{Z}_p^\times \mid a \equiv 1 \pmod{p}\}$

$\underline{p=2}$: $\mu = C_2$, $\mathbb{Z}_2 = \{a \in \mathbb{Z}_2^\times \mid a \equiv 1 \pmod{4}\}$

$\Rightarrow L_{K(1)} S^0 \simeq (K^{h\mu})^{h\mathbb{Z}_p}$.

$\psi \in \mathbb{Z}_p$ "top'l generator" (Adams operation) fiber sequence

$$L_{K(1)}S^0 \longrightarrow K^{h\mu} \xrightarrow{\psi - 1} K^{h\mu} \longrightarrow \Sigma L_{K(1)}S^0$$

example of a <u>finite resolution</u> of $L_{K(1)}S^0$.

- compute $\pi_* K^{h\mu}$ using the homotopy fixed point spectral sequence
- compute $\pi_* L_{K(1)}S^0$ using knowledge of ψ, $\pi_* K^{h\mu}$, & l.e.s. of homotopy groups. //