

G-SPECTRA II: DUALITY, SPECTRA WITH G-ACTION, FIXED POINTS, THE WIRTHMÜLLER ISOMORPHISM, & TOM DIECK SPLITTING.

Duality: Motivation for the category of G-spectra.

Prehistoric observation: (Alexander duality)

$K \subseteq S^n$ compact \Rightarrow

$$\tilde{H}_i(S^n \setminus K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z})$$

" $S^n \setminus K$ is Alexander dual to K ".

Issues: The notion of "dual to K "

- (1) depends on n .
- (2) is not homotopically well-defined (e.g. knots).

Spanier: To resolve (1) \rightarrow first pass at a stable category (1950s) $\rightarrow SC$

$$\text{ob } SC = \text{ob } \text{Top}$$

$$SC(X, Y) = \text{colim}_n \text{Top}(\mathbb{Z}^n X, \mathbb{Z}^n Y)$$

Def: X & Y are Spanier-Whitehead dual if $Y \simeq S^n \setminus X$ in SC .

Formal setting for duality

↳ closed symmetric monoidal category:

Have an adjunction:

$$\text{Hom}(X \wedge Y, Z) \cong \text{Hom}(X, F(Y, Z))$$

with unit = $\text{ev}: X \wedge F(X, Y) \rightarrow Y$

& counit = $\text{coev}: X \rightarrow F(Y, Y \wedge X)$.

Def: X is strongly dualizable with dual Y

if $\exists \epsilon: Y \wedge X \rightarrow S$ & $\eta: S \rightarrow X \wedge Y$ s.t.

$$\left\{ \begin{array}{l} X \cong S \wedge X \xrightarrow{\eta} X \wedge Y \wedge X \xrightarrow{\epsilon} X \wedge S \cong X \\ Y \cong Y \wedge S \xrightarrow{\eta} Y \wedge X \wedge Y \xrightarrow{\epsilon} S \wedge Y \cong Y \end{array} \right.$$

are both the identities.

Prop: X & Y dual \Rightarrow

• $Y \cong F(X, S)$

• $X \cong F(S, X)$.

Duals of manifolds. M compact sm. manifold

Whitney $\Rightarrow M \hookrightarrow \mathbb{R}^n$

Tubular nbhd $\Rightarrow M \hookrightarrow M_\epsilon \hookrightarrow \mathbb{R}^n$
 $\nwarrow \begin{matrix} \text{is} \\ \nu \end{matrix}$

$\Rightarrow \eta: S^n \longrightarrow \mathbb{R}^n / (M_\epsilon^c) = \text{Thom}(\nu) \rightarrow \text{Thom}(\nu) \wedge M_+$

$$\& \quad M \xrightarrow{\Delta} M \times M \xrightarrow{s_0 \times \text{id}} \underbrace{v \times M}_{\text{trivial normal bundle}}$$

$$\text{Pontryagin-Thom} \Rightarrow \text{Thom}(v) \wedge M_+ \longrightarrow \begin{array}{c} \mathbb{Z}^n M_+ \\ \downarrow \\ S^n \wedge M_+ \\ \downarrow \\ S^n \end{array}$$

$\searrow \varepsilon$

Thm: (Atiyah Duality) η & ε exhibit $\text{Thom}(v)$ & M_+ as n -dual in SC .
 \hookrightarrow really $\mathbb{Z}^{-n} \text{Thom}(v)$ & M_+ are dual.
 \Rightarrow Poincaré duality

Moral: if you want a setting for duality, you need to be able to invert spheres.

Equivalently: Whitney embedding: \forall G -mfd M
 \exists G -rep. V & G -equiv. $M \hookrightarrow V$
 \Rightarrow need to invert representation spheres.

The Wirthmüller isomorphism

(Equivalent analogue of $\prod_{i=1}^n X_i \xrightarrow{\sim} \prod_{i=1}^n X_i$)

Thm: $H \in G, X \in Sp^H$. Then $\exists \pi_X^{-1} \cong$
 $G \wedge_H X \xrightarrow{\sim} F_H(G, X)$

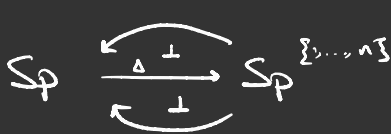
Taking $X = S^0$:

$$\Sigma^\infty (G/H)_+ \cong F_H(G, S^0) \cong F((G/H)_+, S^0)$$

\Rightarrow orbits are self dual.

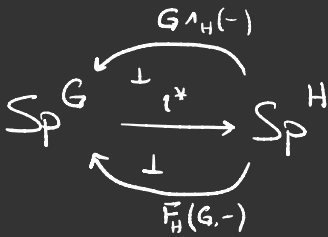
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why analog of $V \rightsquigarrow \Pi$?



left & right adjoints are equivalent.

Δ is "ambidextrous"



Wirthmüller \Rightarrow left & right adjoints equivalent.

i^* is "ambidextrous"



Another approach to G -spectra: Sp^{BG}

$$\left(G \curvearrowright X \right)_{\text{Sp}} \iff \text{functor } BG \rightarrow \text{Sp}$$

One model: Sp^{BG} = orthogonal spectra with a G -action by spectrum maps.

↳ natural notion of weak equivalence given by Borel equivalence: equivariant map of spectra which is an equivalence on underlying spectra.

ISSUE: $X \in \text{Sp}^{BG}$ comes with $G \curvearrowright X(n)$
 $\forall n \in \mathbb{N}$ not $\forall G$ -rep V .

Odd fact: Nevertheless,

$$\begin{array}{ccc} \text{Sp}^{BG} & \begin{array}{c} \xleftarrow{\text{restrict to trivial reps}} \\ \xrightarrow{\text{equivalence of 1-categories}} \end{array} & \text{Sp}^G \\ \downarrow \psi & & \downarrow \psi \\ X & \xrightarrow{\quad} & [V \mapsto \text{O}(\mathbb{R}^n, V)_+ \wedge_{\text{O}(n)} X_n] \end{array}$$

BUT: This equivalence does not preserve weak equivalences.

Let Sp^{bG} be the category Sp^{BG} but endowed with the model structure of Sp^G via the above equivalence.

Then

$$\begin{array}{ccc} & \xrightarrow{\text{restrict to trivial reps}} & \\ & \swarrow & \\ \text{cfl: } Sp^{BG} & \xrightarrow{\perp} & Sp^{bG} \simeq Sp^G \\ \downarrow \omega & & \downarrow \omega \\ X & \longmapsto & F(EG_+, X) \end{array}$$

preserves weak equivalences & is lax symmetric monoidal.

Rmk: cfl is not essentially surjective
i.e. "not every G -spectrum arises as a spectrum with G -action."

e.g. KU_G (Atiyah-Segal)

Def: • $X \in Sp^G$ is cofree if $X \simeq \text{cfl}(\tilde{X})$

$\Leftrightarrow X \simeq \text{cfl}(X|_{\text{trivial}})$

• The cofree completion of X is

$$X^h := \text{cfl}(X|_{\text{trivial}}).$$

Examples:

- Any Borel cohomology theory is cofree
- $K\mathbb{R}$ is cofree
- Atiyah-Segal completion: " $KU_G^h \simeq KU$ "

Fixed points of G -spectra

Three (or four...) ways to construct

$$\mathrm{Sp}^G \times \left\{ \begin{array}{l} \text{subgroups} \\ \text{of } G \end{array} \right\} \longrightarrow \mathrm{Sp}$$

satisfying different properties of what "fixed points" might mean.

Desirable properties of a fixed point functor

- ① In analogy to Elmendorf's Thm, want the collection of all fixed point functors to detect genuine equivalences.
- ② Commute with Σ^∞ (i.e. should agree with fixed points of G -spaces, a.k.a. "geometric")
- ③ Right adjoint to trivial G -action spectrum
↳ Recall: $X \in \mathrm{Set}, Y \in G\text{-set}$.
$$\mathrm{Hom}_G(X, Y) \cong \mathrm{Hom}_{\mathrm{set}}(X, Y^G)$$
- ④a Reduce to fixed points of spectrum w/
 G -action on $\mathrm{Im}(cfl)$
- ④b Have means to calculate (via spectral sequence).

The best we can do is satisfy 2 of these 4,
but we have 3 ways to satisfy different pairs of properties.

I. (Genuine) fixed points (a.k.a. "categorical" when X fibrant)

"Derived space-wise fixed points"

$fX :=$ fibrant replacement of X , $H \leq G$.

$$(X^H)_n := (fX(\mathbb{R}^n))^H \\ \cong F(G/H_+, fX(\mathbb{R}^n))$$

★ Satisfies ① & ③

not ② (explained by tom Dieck splitting)

or ④ (since not all G -spectra are cofree)

II. Homotopy fixed points

"Force ④a by applying cofree completion"

$$X^{hH} := (X^h)^H$$

ie. $(X^{hH})_n = F(EG_+, fX(\mathbb{R}^n))^H$.

★ Satisfies ② & ④

not ① (not all G -spectra are cofree) or ③.

III. Geometric fixed points

"Force ②, strongly sym. monoidal, preserve lby colims"

↳ $\exists!$ such functor, given by

$$\Phi^H(X)_n := X(\mathbb{R}^n \otimes \rho_H)^H$$

regular representation

$$= \operatorname{colim}_{V^H \hookrightarrow \mathbb{R}^n} X(V)^H$$

★ Satisfies ① & ②
not ③ or ④.

Remark: It is possible to satisfy ② & ③ with a notion of fixed points that is not homotopical, namely, categorical fixed points: like genuine, but don't fibrotyly replace first.

Tom Dieck splitting

Above, we said $\Sigma^\infty(X^G) \neq (\Sigma^\infty X)^G$.

Thm: (tom Dieck splitting)

Given $H \leq G$, let $WH = NH/H$. Then

$$(\Sigma^\infty X)^G \simeq \bigvee_{(H) \leq G} \Sigma^\infty EWH_+ \wedge_{WH} X^H$$

$$\pi_*^G(\Sigma^\infty X) \simeq \bigoplus_{(H) \leq G} \pi_*^{WH}(\Sigma^\infty EWH_+ \wedge_{WH} X^H)$$

Cor: $\pi_0^G(\mathbb{S}) \cong \bigoplus_{(H) \leq G} \mathbb{Z}$.