G-SPECTRA II: DUALITY, SPECTRA WITH G-ACTION, FIXED POINTS, THE WIRTHMÜLLER ISOMORPHISM, & TOM DIECK SPLITTING.

Duality: Motivation for the category of G-spectra.

Prehistoric observation: (Alexander duality)

$K \subseteq S^n$ compact $\Rightarrow$

$$\tilde{H}_i(S^n \setminus K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z})$$

"$S^n \setminus K$ is Alexander dual to K".

Issues: The notion of "dual to K"

(1) depends on n.

(2) is not homotopically well-defined (e.g. knots).

Spanier: To resolve (1) $\rightsquigarrow$ first pass at a stable category (1950s) $\to$ SC

$$\mathrm{ob}\, SC = \mathrm{ob}\, Top$$

$$SC(X, Y) = \operatorname{colim}_n Top(\Sigma^n X, \Sigma^n Y)$$

Def: X & Y are Spanier-Whitehead dual if $Y \simeq S^n \setminus X$ in SC.

Formal setting for duality

↳ closed symmetric monoidal category:

Have an adjunction:

$$\mathrm{Hom}(X \wedge Y, Z) \simeq \mathrm{Hom}(X, F(Y,Z))$$

with unit = ev: $X \wedge F(X,Y) \longrightarrow Y$

& counit = coev: $X \longrightarrow F(Y, Y \wedge X)$.

Def: X is <u>strongly dualizable with dual Y</u> if $\exists$ $\varepsilon: Y \wedge X \longrightarrow S$ & $\eta: S \longrightarrow X \wedge Y$ s.t.

$$\begin{cases} X \cong S \wedge X \xrightarrow{\eta} X \wedge Y \wedge X \xrightarrow{\varepsilon} X \wedge S \cong X \\ Y \cong Y \wedge S \xrightarrow{\eta} Y \wedge X \wedge Y \xrightarrow{\varepsilon} S \wedge Y \cong Y \end{cases}$$

are both the identities.

Prop: X & Y dual $\Rightarrow$

- $Y \cong F(X,S)$
- $X \cong F(S,X)$.

<u>Duals of manifolds</u>: M compact sm. manifold

Whitney $\Rightarrow$ $M \hookrightarrow \mathbb{R}^n$

Tubular nbd $\Rightarrow$ $M \hookrightarrow \underset{\underset{\nu}{\cong}}{M_\varepsilon} \hookrightarrow \mathbb{R}^n$

$\Rightarrow \eta: S^n \longrightarrow \mathbb{R}^n/(M_\varepsilon^c) = \mathrm{Thom}(\nu) \longrightarrow \mathrm{Thom}(\nu) \wedge M_+$

$$\& \qquad M \xrightarrow{\Delta} M \times M \xrightarrow{s_0 \times id} \underbrace{\nu}_{\text{trivial normal bundle}} \times M$$

$$\begin{tikzcd}
\text{Pontryagin-Thom} \Rightarrow \mathrm{Thom}(\nu) \wedge M_+ \arrow[r] \arrow[drr, "\varepsilon"'] & \Sigma^n M_+ \\
& S^n \wedge M_+ \arrow[u, phantom, "\cong"] \arrow[d] \\
& S^n
\end{tikzcd}$$

Thm: (Atiyah Duality) η & ε exhibit $\mathrm{Thom}(\nu)$ & M_+ as n-dual in SC.

↳ really $\Sigma^{-n}\mathrm{Thom}(\nu)$ & M_+ are dual.

$\Rightarrow$ Poincare duality

Moral: If you want a setting for duality, you need to be able to invert spheres.

Equivariantly: Whitney embedding: $\forall$ G-mfd M $\exists$ G-rep. V & G-equiv. $M \hookrightarrow V$

$\Rightarrow$ need to invert representation spheres.

The Wirthmüller isomorphism

(Equivariant analogue of $\bigvee_{i=1}^{n} X_i \xrightarrow{\sim} \prod_{i=1}^{n} X_i$)

Thm: $H \leq G$, $X \in Sp^H$. Then $\exists$ π_*-iso
$$G \wedge_H X \xrightarrow{\sim} F_H(G, X)$$

Taking $X = S^0$:

$$\Sigma^\infty (G/H)_+ \cong F_H(G, S^0) \cong F((G/H)_+, S^0)$$

$\Rightarrow$ orbits are self dual.

why analog of $V \xrightarrow{\sim} \Pi$?

$$\begin{tikzcd} Sp \arrow[r, "\Delta"] & Sp^{\{1,\dots,n\}} \arrow[l, bend right, "\perp"'] \arrow[l, bend left, "\perp"] \end{tikzcd}$$

left & right adjoints are equivalent.

i.e. Δ is "ambidextrous"

$$\begin{tikzcd} Sp^G \arrow[r, "i^*"] & Sp^H \arrow[l, bend right, "G \wedge_H (-)"', "\perp"] \arrow[l, bend left, "F_H(G,-)", "\perp"'] \end{tikzcd}$$

Wirthmüller $\Rightarrow$ left & right adjoints equivalent.

i.e. i^* is "ambidextrous".

Another approach to G-spectra: Sp^{BG}

$$\left(G \circlearrowright X\right) \iff \text{functor } BG \longrightarrow Sp\,.$$

$\rotatebox{90}{$\in$}$ Sp

<u>One model</u>: Sp^{BG} = orthogonal spectra with a G-action by spectrum maps.

↳ natural notion of weak equivalence given by <u>Borel equivalence</u>: equivariant map of spectra which is an equivalence on underlying spectra.

<u>ISSUE</u>: $X \in Sp^{BG}$ comes with $G \circlearrowright X(n)$ $\forall n \in \mathbb{N}$ not $\forall$ G-rep V.

<u>Odd fact</u>: Nevertheless,

$$\begin{tikzcd} Sp^{BG} \arrow[rr, bend right, "\sim"', "\text{equivalence of 1-categories}"] & & Sp^{G} \arrow[ll, bend right, "\text{restrict to trivial reps}"'] \\ X \arrow[rr, mapsto] & & \left[V \mapsto O(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n\right] \end{tikzcd}$$

<u>BUT</u>: This equivalence does not preserve weak equivalences.

Let Sp^{bG} be the category Sp^{BG} but endowed with the model structure of Sp^G via the above equivalence.

Then

$$\begin{tikzcd} \mathrm{cfl}: Sp^{BG} \arrow[r, "\perp"] & Sp^{bG} \arrow[r, phantom, "\simeq"] & Sp^{G} \arrow[ll, bend right, "\text{restrict to trivial reps}"'] \\ X \arrow[r, mapsto] & F(EG_+, X) & \end{tikzcd}$$

preserves weak equivalences & is lax symmetric monoidal.

Rmk: cfl is not essentially surjective

i.e. "not every G-spectrum arises as a spectrum with G-action."

e.g. KU_G (Atiyah-Segal)

Def: • $X \in Sp^G$ is <u>cofree</u> if $X \simeq \mathrm{cfl}(\tilde{X})$

$\Longleftrightarrow$ $X \simeq \mathrm{cfl}(X|_{\text{trivial}})$

• The <u>cofree completion</u> of X is

$$X^h := \mathrm{cfl}(X|_{\text{trivial}}).$$

<u>Examples</u>:

- Any Borel cohomology theory is cofree
- $K\mathbb{R}$ is cofree
- Atiyah-Segal completion: "$KU_G^h \simeq KU$"

Fixed points of G-spectra

Three (or four...) ways to construct

$$Sp^G \times \left\{ \begin{matrix} \text{subgroups} \\ \text{of } G \end{matrix} \right\} \longrightarrow Sp$$

satisfying different properties of what "fixed points" might mean.

Desirable properties of a fixed point functor

① In analogy to Elmendorf's Thm, want the collection of all fixed point functors to detect genuine equivalences.

② Commute with Σ^∞ (i.e. should agree with fixed points of G-spaces, a.k.a. "geometric")

③ Right adjoint to trivial G-action spectrum

↳ Recall: $X \in \mathrm{Set}$, $Y \in G\text{-Set}$.

$$\mathrm{Hom}_G(X, Y) \cong \mathrm{Hom}_{\mathrm{Set}}(X, Y^G)$$

4a Reduce to fixed points of spectrum w/ G-action on Im (cf ℓ)

4b Have means to calculate (via spectral sequence).

The best we can do is satisfy $\underline{2}$ of these 4, but we have 3 ways to satisfy different pairs of properties.

I. (Genuine) fixed points (a.k.a. "categorical" when X fibrant)

"Derived space-wise fixed points".

$fX :=$ fibrant replacement of X, $H \leq G$.

$$(X^H)_n := (fX(\mathbb{R}^n))^H$$
$$\simeq F(G/H_+, fX(\mathbb{R}^n))$$

★ Satisfies ① & ③

not ② (explained by tom Dieck splitting)

or ④ (since not all G-spectra are cofree)

II. Homotopy fixed points

"Force ④a by applying cofree completion".

$$X^{hH} := (X^h)^H$$

i.e. $(X^{hH})_n = F(EG_+, fX(\mathbb{R}^n))^H$.

★ Satisfies ② & ④

not ① (not all G-spectra are cofree) or ③.

III. Geometric fixed points

"Force ②, strong sym. monoidal, preserve hty colims"

↳ ∃! such functor, given by

regular representation

$$\Phi^H(X)_n := X(\mathbb{R}^n \otimes \rho_H)^H$$

$$= \operatorname{colim}_{V^H \hookrightarrow \mathbb{R}^n} X(V)^H$$

★ Satisfies ① & ②

not ③ or ④.

Remark: It is possible to satisfy ② & ③ with a notion of fixed points that is not homotopical, namely, categorical fixed points: like genuine, but don't fibrantly replace first.

Tom Dieck splitting

Above, we said $\Sigma^\infty(X^G) \neq (\Sigma^\infty X)^G$.

Thm: (tom Dieck splitting)

Given $H \leq G$, let $WH = NH/H$. Then

$$(\Sigma^\infty X)^G \simeq \bigvee_{(H) \leq G} \Sigma^\infty EWH_+ \wedge_{WH} X^H$$

$$\pi_*^G(\Sigma^\infty X) \simeq \bigoplus_{(H) \leq G} \pi_*^{WH}(\Sigma^\infty EWH_+ \wedge_{WH} X^H)$$

Cor: $\pi_0^G(\mathbb{S}) \cong \bigoplus_{(H) \leq G} \mathbb{Z}$.