Equivariant K-theory

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Most of the content in this talk can be found in Segal's thesis *Equivariant K-theory* [4]. I also learned some of the material from Atiyah's *K-theory* [1].

1 Motivation

Recall, if $(A, \oplus)$ is a commutative monoid, then

$$K(A) := F(A)/\langle a + b - a \oplus b \rangle$$

is an abelian group. When A is also a semiring, $K(A)$ is a ring.

Last time, we introduced the *representation ring* of a group G, defined as

$$R(G) := K(\mathrm{Rep}(G)),$$

where $\mathrm{Rep}(G)$ is the semiring of isomorphism classes of representations of G under direct sum and tensor product. Similarly, the *K-theory* of a space X is

$$K(X) := K(\mathrm{Vect}(X)),$$

where $\mathrm{Vect}(X)$ is the semiring of isomorphism classes of complex vector bundles over X under direct sum and tensor product. There is a *classifying space* functor $B : \mathsf{Group} \to \mathsf{Top}$, that associates to a group G, a space BG that classifies principal G-bundles. It is natural to ask how the two functors R and K compare after passing through B. In other words,

Question 1.1. *What is the relationship between $R(G)$ and $K(BG)$?*

We can start to make progress on this question by considering the universal G-bundle

$$\pi : EG \to BG.$$

Given a representation $\rho : G \to GL(\mathbb{C}^n)$, we can form the vector bundle

$$\pi_\rho : EG \times_G \mathbb{C}^n \to BG$$

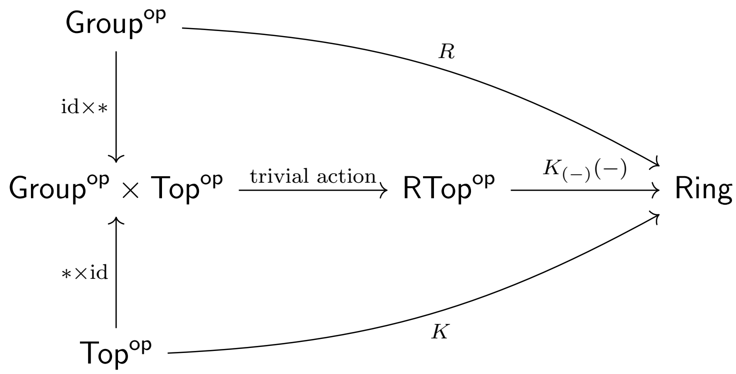
as the quotient by the diagonal action of G on $EG \times \mathbb{C}^n$. The association $\rho \mapsto \pi_\rho$ induces a ring homomorphism

$$R(G) \to K(BG).$$

In order to go further, we'll want to combine R and K into a single functor that contains both representation theoretic and topological information. A convenient domain for such a functor is the category, RTop, of group actions on spaces. The objects of RTop are G-spaces, for all groups G, and the morphisms are group homomorphisms, equivariant continuous maps, and their compositions.

2 Equivariant K-theory

As indicated above, we seek a functor $K_{(-)}(-) : \mathsf{RTop}^{\mathsf{op}} \to \mathsf{Ring}$, called *equivariant K-theory*, which fits into the following diagram.



The value of $K_{(-)}(-)$ on a G-space X is written $K_G X$, suppressing the action from the notation, and the above diagram commuting means that

$$K_G(X) = \begin{cases} R(G), & X = * \\ K(X), & G = *. \end{cases}$$

In fact, when restricting to the category of trivial actions, we'll have $K_G(X) \cong R(G) \otimes K(X)$.

Definition 2.1. Let X be a G-space. A *G-vector bundle* over X is a G-map $p : E \to X$, such that

- p has the structure of a vector bundle;
- for all $g \in G$, the map $g : E_x \to E_{gx}$ is linear.

Example 2.2. If X is a smooth manifold and G acts smoothly on X, then $TX \otimes \mathbb{C}$ is a G-vector bundle on X.

Example 2.3. If $E \to X$ is an ordinary vector bundle, then $E^{\otimes n} := E \otimes \cdots \otimes E \to X$ is a Σ_n-vector bundle, where Σ_n permutes the factors of $E^{\otimes n}$ and acts trivially on X.

Notice that $\mathrm{Vect}_G(X) := \{\text{isomorphism classes of } G\text{-vector bundles over } X\}$ is a semiring under direct sum and tensor product.

Definition 2.4. The *equivariant K-theory* of a G-space X is

$$K_G(X) := K(\mathrm{Vect}_G(X)).$$

Remark 2.5 (functoriality)**.** If $f : X \to Y$ is a G-map of G-spaces, then pullback induces a map $f^* : K_G(Y) \to K_G(X)$. Similarly, a homomorphism of groups $\varphi : H \to G$ induces a map $\varphi^* : K_G(X) \to K_H(X)$ by restricting a G-action to H via φ. Under these assignments, $K_{(-)}(-)$ is a functor.

Remark 2.6 (homotopy invariance)**.** If $f_0, f_1 : X \to Y$ are G-homotopic (i.e. homotopic via a homotopy that is also a G-map, where the action on $X \times I$ is induced by the trivial action on I), then $f_0^* = f_1^*$.

Example 2.7. If G is the trivial group, then a G-vector bundle is the same as a vector bundle, so $K_G(X) = K(X)$.

Example 2.8. If X is a one point space, then a vector bundle is the same as a vector space. Thus, a G-vector bundle is a representation of G, so $K_G(X) = R(G)$.

Example 2.9. If X is a trivial G-space, then the action of G on the total space of a G-vector bundle restricts to actions on each fiber. This leads to two apparent inclusions,

$$i : \mathrm{Rep}(G) \xrightarrow{\text{trivial bundle}} \mathrm{Vect}_G(X)$$

$$j : \mathrm{Vect}(X) \xrightarrow[\text{trivial action}]{} \mathrm{Vect}_G(X)$$

where i takes a representation M, to the trivial bundle $\mathbf{M} := X \times M$, and j endows a vector bundle E with a trivial G-action.

Theorem 2.10. *Let X be a trivial G-space. The map $\tilde{i} \otimes \tilde{j} : R(G) \otimes K(X) \to K_G(X)$ induced by the inclusions i and j above, is an isomorphism of rings.*

Proof idea: To define an inverse $\nu : K_G(X) \to R(G) \otimes K(X)$, note that each fiber E_x of a G-vector bundle E on X is a representation of G, and can therefore be expressed as a linear combination of irreducible representations

$$E_x \cong \bigoplus_{\text{irrep's } M} M \otimes \mathrm{Hom}_G(M, E_x).$$

Following this, we define

$$\nu(E) = \bigoplus_{\text{irrep's } M} \mathbf{M} \otimes \mathrm{Hom}_G(\mathbf{M}, E).$$

The fact that ν is the inverse of $\tilde{i} \otimes \tilde{j}$ follows from the fact that a fiberwise isomorphism of vector bundles is an isomorphism.

Example 2.11. In Example 2.3, we constructed a map $K(X) \to K_{\Sigma_n}(X)$ given by the nth tensor power of a vector bundle. In light of Theorem 2.10, this gives a homomorphism

$$K(X) \to R(\Sigma_n) \otimes K(X).$$

A choice of homomorphism $R(\Sigma_n) \to \mathbb{Z}$ results in an endomorphism of $K(X)$. Since this construction is natural in X, we get a map

$$\mathrm{Hom}(R(\Sigma_n), \mathbb{Z}) \to \mathrm{Op}(K).$$

The image of this map consists of the power operations in K-theory.

3 K_G as equivariant cohomology

We start by indicating that, like K-theory, equivariant K-theory deserves to be considered a cohomology theory.

Definition 3.1. Let $(X, *)$ be a pointed G-space, and $A \subset X$ a closed G-subspace.

- $\tilde{K}_G(X) :=$ $\ker(K_G(X) \xrightarrow{i^*} K_G(*))$, where $i : * \to X$ is the inclusion;
- $\tilde{K}_G^{-q}(X) := \tilde{K}_G(S^q X)$;
- $\tilde{K}_G^{-q}(X, A) := \tilde{K}_G(S^q(X \coprod_A CA))$.

When X is not pointed, define

- $K_G^{-q}(X) := \tilde{K}_G^{-q}(X \coprod *)$;
- $K_G^{-q}(X, A) := \tilde{K}_G^{-q}(X \coprod *, A \coprod *)$.

With these definitions, we have a long exact sequence

$$\cdots \to \tilde{K}_G^{-q}(X, A) \to \tilde{K}_G^{-q}(X) \to \tilde{K}_G^{-q}(A) \to \tilde{K}_G^{-q+1}(X, A) \to \cdots.$$

Example 3.2. We know that $K^0_{S^1}(*) = K_{S^1}(*) \cong R(S^1) \cong \mathbb{Z}[x, x^{-1}]$. In degree -1:

$$K^{-1}_{S^1}(*) \cong \tilde{K}^{-1}_{S^1}(S^0) \cong \tilde{K}_{S^1}(S^1) = \ker(K_{S^1}(S^1) \to K_{S^1}(*)).$$

Here, the action of S^1 on S^1 is trivial, since it is induced by the suspension of S^0. By Theorem 2.10,

$$K_{S^1}(S^1) \cong R(S^1) \otimes K(S^1) \cong R(S^1) \otimes (\mathbb{Z} \oplus (\mathbb{Z}/2)) \cong R(S^1) \oplus (R(S^1) \otimes \mathbb{Z}/2) \cong R(S^1),$$

since $R(S^1) \otimes \mathbb{Z}/2 \cong \mathbb{Z}[x, x^{-1}] \otimes \mathbb{Z}/2 = 0$. Thus, the restriction map

$$R(G) \cong K_{S^1}(S^1) \to K_{S^1}(*) \cong R(G)$$

is an isomorphism, and hence has no kernel. So $K^{-1}_{S^1}(*) = 0$.

The following theorem justifies regarding K_G as the G-equivariant verison of the cohomology theory K.

Theorem 3.3. *Suppose G acts on X freely. Then*

$$K_G(X) \cong K(X/G)$$

Example 3.4. For any Lie group G, $K_G(G) \cong K(*) \cong \mathbb{Z}$. More generally, if $H \leq G$ is a closed subgroup, then $K_G(G/H) \cong K_H(*) \cong R(H)$.

Theorem 3.3 indicates that the relationship between K_G and K is similar to the relationship between ordinary (Borel) equivariant cohomology H_G^* and ordinary cohomology H^*. In the ordinary case, when the action of G on X is free, we define

$$H_G^*(X) := H^*(X/G).$$

When the action is not free, we still use this idea to define $H_G^*(X)$, by replacing X with a (homotopy equivalent) G-space for which the action is free. More specifically, consider the total space EG of the universal bundle over BG. Then EG is contractible and is a free G-space. Define

$$X_G := EG \times_G X = (EG \times X)/G.$$

The space X_G acts as a replacement for X/G, and we define

$$H_G^*(X) := H^*(X_G).$$

In analogy with the ordinary case, we can consider the ring $K^*(X_G)$ and ask how it compares with our definition of equivariant K-theory.

Question 3.5. *What is the relationship between $K_G^*(X)$ and $K^*(X_G)$?*

As in our initial discussion of Question 1.1, we can construct a map $\mathrm{Vect}_G(X) \to \mathrm{Vect}(X_G)$, as follows: A G-vector bundle $E \to X$ comes with an action of G, so we can apply $- \times_G EG$ to both E and X to get a vector bundle $E \times_G EG \to X_G$.

Theorem 3.6 (Atiyah-Segal [3])**.** *If $K_G^*(X)$ is a finitely generated $R(G)$-module, the map $K_G^*(X) \to K^*(X_G)$ induces an isomorphism*

$$\lim_n \left(K_G^*(X)/I_G^n \cdot K_G^*(X) \right) \xrightarrow{\sim} K^*(X_G),$$

where I_G is the kernel of the degree homomorphism $R(G) \to \mathbb{Z}$ induced by the dimension of a representation.

The ideal I_G is also known as the *augmentation ideal* of $R(G)$, and the limit of quotients $K_G^*(X)_{\hat{I}_G} := \lim_n(K_G^*(X)/I_G^n \cdot K_G^*(X))$ is called the *completion of $K_G^*(X)$ with respect to the augmentation ideal*. Since $K_G^*(X)_{\hat{I}_G}$ can be recovered from $K_G^*(X)$, we see that $K^*(X_G)$ contains only part of the information that $K_G^*(X)$ does.

Example 3.7. Let $X = *$. Then $K_G^*(X) = R(G)$ and $X_G = EG/G \cong BG$, so

$$K^*(BG) \cong R(G)_{\hat{I}_G},$$

answering Question 1.1.

4 The Thom isomorphism and Bott periodicity

Recall [2] that one way to describe classes in relative K-theory is with chain complexes of vector bundles. Specifically,

$$K(X, A) \cong L(X, A)/\sim$$

where $L(X, A)$ is the set of chain complexes of vector bundles over X which are acyclic except on a compact set contained in $X \setminus A$, and $\sim$ is a certain type of homotopy relation. The situation is no different in the G-equivariant setting: we can write

$$K_G(X, A) \cong L_G(X, A)/\sim$$

where the chain complexes in $L_G(X, Y)$ consist of G-vector bundles. We will use this chain complex representation of K_G to define the Thom isomorphism.

Given a G-vector bundle $p : E \to X$, we can form the pullback of E along p to get a G-vector bundle p^*E over E:

$$\begin{array}{ccc} p^*E & \longrightarrow & E \\ \downarrow & & \downarrow p \\ E & \xrightarrow[p]{} & X. \end{array}$$

There is a canonical section $\delta : E \to p^*E$ given by the diagonal. We can form a chain complex

$$\Lambda^\bullet_E = \cdots \to 0 \to \mathbb{C} \xrightarrow{d} \Lambda^1 p^*E \xrightarrow{d} \Lambda^2 p^*E \to \cdots$$

where d is defined on $\Lambda^i p^*E_x$ by $d(\xi) = \xi \wedge \delta(x)$. Then $\Lambda^\bullet_E$ represents a class $[\Lambda^\bullet_E] \in K_G(E)$, and we can define a map

$$\mathrm{Th} : K_G(X) \to K_G(E)$$

called the *Thom homomorphism* by

$$\mathrm{Th}([F^\bullet]) = [\Lambda^\bullet_E \otimes p^*F^\bullet].$$

Theorem 4.1 (Thom isomorphism [4])**.** *For any G-vector bundle E on a locally compact G-space X, the Thom homomorphism*

$$\mathrm{Th} : K_G(X) \to K_G(E)$$

is an isomorphism.

Corollary 4.2 (Bott periodicity)**.** *Under the assumptions of Theorem 4.1,*

$$K_G^{-q}(X) \cong K_G^{-q-2}(X).$$

To see this, apply Theorem 4.1 to the trivial bundle $X \times \mathbb{C}$. The degree shift is by 2 due to the fact that $\mathbb{C}$ has 2 real dimensions.

References

[1] Atiyah, M. F., *K-theory.*

[2] Atiyah, M. F., Bott, R., and Shapiro, A., *Clifford modules.*

[3] Atiyah, M. F., Segal, G., *Equivariant K-theory and completion.*

[4] Segal, G., *Equivariant K-theory.*