Clifford algebras, modules, & Morita theory

The objects of interest

Def: Given a real vector space V with a quadratic form q on V, define the Clifford algebra of (V, q) as the $\mathbb{R}$-algebra

$$Cl(V, q) := T(V) \big/ (v \otimes v + q(v))$$

Ex: Let $\mathbb{R}^{p|q}$ be the vector space $\mathbb{R}^{p+q}$ with bilinear form $b(v, w) =$

$$b\left(\sum_{i=1}^{p+q} v_i e_i, \sum_{j=1}^{p+q} w_j e_j\right) = \sum_{i=1}^{p} v_i w_i - \sum_{i=p+1}^{p+q} v_i w_i .$$

Notation:

- $Cl_{p,q} := Cl(\mathbb{R}^{p|q})$
- $Cl_n := Cl_{n,0}$
- $Cl_{-n} := Cl_{0,n}$

A generators & relations presentation

$$Cl_{p,q} \cong \left\langle \begin{matrix} f_1, \dots, f_p, \\ e_1, \dots, e_q \end{matrix} \;\middle|\; \begin{matrix} f_i f_j + f_j f_i = -2\delta_{ij} \\ e_i e_j + e_j e_i = 2\delta_{ij} \\ f_i e_j + e_j f_i = 0 \end{matrix} \right\rangle$$

Remarks:

- distinct generators anti-commute while the first p square to -1 & the last q square to 1.

- as a vector space, $Cl_{p,q} \cong \Lambda^\bullet \mathbb{R}^{p+q}$, where the multiplication is "deformed" away from the super-commutative multiplication of the exterior algebra. In fact,

$$\Lambda^\bullet \mathbb{R}^n \cong Cl(\mathbb{R}^n, q=0) \quad \text{as algebras.}$$

Examples:

① Cl_0 is an $\mathbb{R}$-algebra with no generators

$$\Rightarrow Cl_0 \cong \mathbb{R}.$$

② Cl_1 is an $\mathbb{R}$-algebra with one generator

f_1 that squares to -1

$\Rightarrow Cl_1 \cong \mathbb{C}$.

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(3) Cl_{-1} : 1 generator $e_1^2 = 1$

$$\Rightarrow Cl_{-1} \cong \mathbb{R}[x]/(x^2-1) \not\cong Cl_1$$

(e.g. $\sqrt{-1} \notin Cl_{-1}$) .

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(4) Cl_2 : as a vector space $= \mathbb{R}\{f_1, f_2, f_1f_2\}$

$f_1^2 = -1, \quad f_2^2 = -1,$

$$(f_1f_2)^2 = f_1f_2f_1f_2 = -f_1^2f_2^2 = -1$$

$\Rightarrow Cl_2 \cong \mathbb{H}$.

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(5) $Cl_{1,1} \xrightarrow{\sim} M_2(\mathbb{R})$

$$f_1 \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$e_1 \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is an isomorphism of $\mathbb{R}$-algebras.

The symmetric monoidal category SVect

Much of the structure of Clifford algebras is best described in terms of their natural $\mathbb{Z}/2$-grading.

<u>Def:</u> Let SVect (super vector spaces) be the category with

- ob(SVect): $\mathbb{Z}/2$-graded $\mathbb{R}$-vector spaces
- mor(SVect): grading preserving linear maps.
- monoidal structure: $V = V^0 \oplus V^1$, $W = W^0 \oplus W^1$,

$$(V \otimes W)^0 = V^0 \otimes W^0 \oplus V^1 \otimes W^1$$

$$(V \otimes W)^1 = V^0 \otimes W^1 \oplus V^1 \otimes W^0$$

- symmetric structure: $V \otimes W \to W \otimes V$

$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$

Some properties of SVect

- SVect is a closed symmetric monoidal category with internal hom:

  $\underline{\mathrm{Hom}}(V,W) = \{\text{all linear maps } V \to W\}$

  graded by whether a map preserves or reverses the grading.

  i.e. $\mathrm{Hom}(V,W) = \underline{\mathrm{Hom}}(V,W)^0$.

- SVect has an enriched structure over itself which we'll denote $\underline{\mathrm{SVect}}$.

- $Cl_{p,q}$ is an algebra object in SVect with $\mathbb{Z}/2$-grading induced from $T(\mathbb{R}^{p+q})$.

- $\Lambda^\bullet(V)$ is a commutative algebra in SVect,

  i.e. $\Lambda^\bullet(V)^{op} \cong \Lambda^\bullet(V)$ ← this uses the symmetric structure of SVect.

- $Cl_n^{op} \cong Cl_{-n}$ ☆ This relation can be refined

- For $n,m \geq 0$: $Cl_n \otimes Cl_m \cong Cl_{n+m}$

- For $n,m \geq 0$: $Cl_n \otimes Cl_{-m} \cong Cl_{n,m}$

Given an algebra object A, define ${}_A\mathrm{Mod}$ & Mod_A as the categories of left & right A-modules, respectively. Similarly, ${}_A\mathrm{Mod}_B$ = category of A-B bimodules. Then

$$\begin{cases} {}_{C\ell_n}\mathrm{Mod} \simeq \mathrm{Mod}_{C\ell_{-n}} & \& \\ {}_{C\ell_n \otimes C\ell_{-m}}\mathrm{Mod} \simeq {}_{C\ell_n}\mathrm{Mod}_{C\ell_m} \simeq \mathrm{Mod}_{C\ell_{-n}\otimes C\ell_m} \end{cases}$$

<u>Remarks</u>:

- Since the categories above use module objects in SVect, objects are $\mathbb{Z}/2$-graded modules & morphisms are grading preserving module maps. Using $\underline{\mathrm{SVect}}$ instead results in categories ${}_A\underline{\mathrm{Mod}}$, $\underline{\mathrm{Mod}}_A$, & ${}_A\underline{\mathrm{Mod}}_B$ with similar relations.
- Due to the above, we can exchange a left $C\ell_n$-module for a right $C\ell_{-n}$-module:

$$m \cdot_r a = (-1)^{|a||m|} a \cdot_\ell m .$$

<u>Ex</u>: The isomorphism $C\ell_{1,1} \longrightarrow M_2(\mathbb{R}) \cong \underline{\mathrm{End}}(\mathbb{R}^{1|1})$ realizes $\mathbb{R}^{1|1}$ as a left $C\ell_{1,1}$-module, or as a $C\ell_1$-$C\ell_1$ bimodule. Note the $\underline{\mathrm{End}}$ & not End.

Morita theory

Review the situation for ordinary rings.

Def: Let BRing denote the 2-category with rings as objects, bimodules as 1-morphisms, & bimodule maps as 2-morphisms.
A *Morita equivalence* between rings R & S is an isomorphism $R \longrightarrow S$ in BRing.

i.e. A bimodule ${}_R M_S$ s.t. $\exists$ a bimodule ${}_S N_R$ with ${}_R M \otimes_S N_R \cong {}_R R_R$ & ${}_S N \otimes_R M_S \cong {}_S S_S$.

Clearly, a Morita equivalence induces an equivalence of categories ${}_R\mathrm{Mod} \simeq {}_S\mathrm{Mod}$.

The converse is also true.

Thm: (Morita) If $F : {}_R\mathrm{Mod} \xrightarrow{\sim} {}_S\mathrm{Mod}$, then $\exists$ Morita equivalence ${}_R M_S$.

Pf sketch: $F({}_R R)$ is a left S-module. For $r \in R$, right mult. is a morphism $m_r : R \to R$. Define $F({}_R R) \circlearrowleft R$ by $x \cdot r = F(m_r)(x)$. //

What about in the graded case?

<u>Example:</u> Recall the bimodule ${}_{Cl_1}\mathbb{R}^{1|1}{}_{Cl_1} \cong {}_{Cl_{1,1}}\mathbb{R}^{1|1}{}_{\mathbb{R}}$

$$\Rightarrow Cl_{1,1} \underset{\text{Morita}}{\simeq} Cl_0$$

$$\Rightarrow Cl_n \otimes Cl_m \underset{\text{Morita}}{\simeq} Cl_{n+m} \quad \forall n, m \in \mathbb{Z}.$$

<u>Example:</u> Let $\mathbb{O} \cong \mathbb{R}^8$ be the octonions. If $x \in \mathbb{R}^8 \subseteq Cl_8$, let $x_{\mathbb{O}} \in \mathbb{O}$ be the corresponding element. Viewing $\mathbb{R}^{8|8}$ as $\mathbb{O} \oplus \mathbb{O}$, we define $\mathbb{R}^8 \longrightarrow \underline{\mathrm{End}}(\mathbb{R}^{8|8})$ by

$$x \longmapsto \begin{bmatrix} 0 & x_{\mathbb{O}} \\ -\bar{x}_{\mathbb{O}} & 0 \end{bmatrix}.$$

This extends to an isomorphism of super algebras

$$Cl_8 \cong \underline{\mathrm{End}}(\mathbb{R}^{8|8}) \Longrightarrow Cl_n \underset{\text{Morita}}{\simeq} Cl_{n+8} \quad \forall n \in \mathbb{Z}$$

<u>FACT:</u> $\nexists$ Morita equivalence ${}_{Cl_n}M_{Cl_m}$, for $|n-m| < 8$.

Before addressing the analog of Morita theory in the graded case, let's take a quick detour.

Addressing varying Clifford sign conventions

Define a lax monoidal functor

$F : \mathrm{SVect} \longrightarrow \mathrm{SVect}$ given by the

identity functor, with lax structures

unit : $\mathbb{R}^{1|0} \longrightarrow \mathbb{R}^{1|0}$ identity

mult : $V \otimes W \longrightarrow V \otimes W$

$v \otimes w \longmapsto (-1)^{|v||w|} v \otimes w$

$\Rightarrow$ F sends algebras to algebras, in fact

$$F(\mathcal{C}\ell_n) \cong \mathcal{C}\ell_{-n}$$

Additionally, F is a strong monoidal equivalence

$\Rightarrow$ F preserves categories of modules

$$\Rightarrow \quad {}_{\mathcal{C}\ell_n}\mathrm{Mod} \simeq {}_{\mathcal{C}\ell_{-n}}\mathrm{Mod} \quad !$$

This functor F is why two presentations of a theory involving Clifford modules using <u>different</u> sign conventions can get results with the <u>same</u> signs!

What happened?

Recall, the construction of the Morita bimodule ${}_S F({}_R R)_R$ relied on the fact that right multiplication by $r \in R$ is a morphism $R \longrightarrow R$ in ${}_R\mathrm{Mod}$.

super algebra

$$a \in A \text{ odd} \implies m_a \text{ is odd}$$

$$\implies m_a \notin \mathrm{Hom}(A, A)$$

so ${}_A\mathrm{Mod} \simeq {}_B\mathrm{Mod} \not\Longrightarrow A \underset{\text{Morita}}{\simeq} B$

However, $\underline{\mathrm{Hom}}(A, A)$ contains odd maps, so

Thm: if A & B are super algebras, then ${}_A\underline{\mathrm{Mod}} \simeq {}_B\underline{\mathrm{Mod}} \iff A \underset{\text{Morita}}{\simeq} B$

Thm: $\left(\{ Cl_n \mid n \in \mathbb{Z} \}, \otimes \right) \Big/ \text{Morita equivalence}$ $\cong \mathbb{Z}/8$. This is the <u>super Brauer group</u> of $\mathbb{R}$.