Bousfield localization
Chromatic homotopy theory reading group
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1 Localization

Definition 1. (7.1.1 E_*-acyclic, E_*-equivalence, E_*-local) A spectrum X is E_*-*acyclic* if $E_*X = 0$. A map $f : X \to Y$ is an E_*-*equivalence* if it induces an isomorphism in E_*-homology. A spectrum Y is E_*-*local* if for each E_*-*acyclic* spectrum X, we have $[X, Y] = 0$.

Definition 2. (7.1.1 E_*-localization) An E_*-localization functor L_E is a covariant functor that takes a spectrum X to a E_*-local spectrum L_EX along with a natural transformation $\eta : id_S \to L_E$ such that

1. $\eta_X : X \to L_EX$ is an E_*-equivalence.
2. For any E_*-equivalence $f : X \to Y$, there is $g : Y \to L_EX$ such that $gf = \eta_X$. In other words, η_X is the terminal E_*-equivalence from X.

Proposition 3. *(7.1.2 Properties of E_*-locality)*

1. *Any inverse limit preserves E_*-locality.*
2. *In a cofiber sequence $W \to X \to Y$, if any two are E_*-local, so is the third.*
3. *Wedge summands preserve E_*-locality, i. e. if $X \vee Y$ is E_*-local, so are X or Y.*

Proposition 4. *(Properties of L_E)*

1. *L_E is unique.*
2. *L_E is idempotent, i. e. $L_EL_E = L_E$*
3. *Any $f : X \to Y$ where Y is E_*-local factors through L_EX. In other words, L_EX is the initial object that is E_*-local and mapped to from X.*

Proposition 5. *Let E_* and F_* be generalized homology theories such that $E_*X = 0$ implies $F_*X = 0$. Then if a spectrum Y is F_*-local, it is E_*-local. In particular, if $E_*X = 0$ iff $F_*X = 0$, then the functors L_E and L_F are the same.*

Theorem 6. *(7.1.3 Bousfield localization) For any homology theory E_* and any spectrum X, the localization L_EX exists and is functorial in X.*

Example 7. (Ravenel 1.9, 1.10) Note that direct limit does not necessarily preserve E_*-locality and L_E does not necessarily commute with inverse limit. See Ravenel's paper for example using the Moore spectrum.

Example 8. (7.14) Let E_* be the ordinary homology H_* and X a finite spectrum satisfying $K(n)_*(X) \neq 0$ with a v_n-self map f. Define the telescope of the system

$$\hat{X} := \lim_{\to}(X \xrightarrow{f} \Sigma^{-d}X \xrightarrow{f} \Sigma^{-2d}X \xrightarrow{f} \cdots).$$

We know $L_H\hat{X}$ is contractible since $H_*(f) = 0$ and therefore $H_*(\hat{X}) = 0$. On the other hand, we know $\hat{X}$ is not contractable because

$$K(n)_*(\hat{X}) \cong K(n)_*(X) \neq 0$$

by properties of v_n-self map.

Proposition 9. *(7.1.5 Ring spectra) If E is a ring spectrum, then any E-module spectrum M (e.g. the spectrum $E \wedge X$ for any X) is E_*-local. Generally, if M is a E-module spectrum, then M is E_*-local.*

Proof. By definition, we need to show that for any spectrum W with $E_*(W) = 0$,

$$[W, E \wedge X] = 0.$$

Let $\mu : E \times E \to E$ denote the multiplication map. Given any map $f : W \to E \wedge X$, we have a diagram

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "\eta\wedge W"'] & E \wedge X \arrow[d, "\eta\wedge X"'] \arrow[dr, "E\wedge X"] & \\ E \wedge W \arrow[r, "E\wedge f"] & E \wedge E \wedge X \arrow[r, "\mu\wedge X"] & E \wedge X. \end{tikzcd}$$

Since $E \wedge W$ is contractable, then f is null. □

Note that this does not imply that $E \wedge X = L_E X$ even though $E \wedge X$ is E_*-local.

Definition 10. (7.1.6 E_*-nilpotence) For a ring spectrum E, the class of *E-nilpotent* spectra is the smallest class satisfying the following conditions:

1. E is E-nilpotent.
2. If N is E-nilpotent, so is $N \wedge X$ for any spectrum X.
3. The cofiber of any map between E-nilpotent spectra is E-nilpotent.
4. Any retract of an E-nilpotent spectrum is E-nilpotent.

Proposition 11. *(7.1.7 E_*-ilpotence and E_*-locality) Every E_* nilpotent spectrum is E_*-local.*

2 Bousfield localization

Definition 12. (7.2.1 Bousfield equivalent) For a spectrum E, $< E >$ denotes the equivalence class of E under the following equivalence relation: $E \sim F$ if $E_*X = 0$ iff $F_*X = 0$ for any spectrum X (i. e. $E \wedge X$ is contractable if and only if $F \wedge X$ is contractable). Equivalently, $E \sim F$ if a map is an E_*- equivalence iff it is a F_*-equivalence. We will refer to $< E >$ as the Bousfield class of E.

Definition 13. (complement) $< F > \geq < E >$ if each F_*-acyclic spectrum is E_*-acyclic. $< E >>< F >$ if $< E > \geq < F >$ and $< E > \neq < F >$. A class $< E >$ has a complement $< E >^c$ if $< E > \wedge < E >^c = < pt >$ and $< E > \vee < E >^c = < S^0 >$.

Proposition 14. *(Properties)*

1. $< E \vee F > = < E > \vee < F >$
2. $< E \wedge F > = < E > \wedge < F >$
3. $(< X > \vee < Y >) \wedge < Z > = (< X > \vee < Z >) \wedge (< Y > \vee < Z >)$
4. $(< X > \wedge < Y >) \vee < Z > = (< X > \wedge < Z >) \vee (< Y > \wedge < Z >)$

Proposition 15. *(Simple examples)*

1. $< S^0 > \geq < E > \geq < pt >$ *for any spectrum* E
2. $< S^0 > \wedge < E > = < E >$
3. $< S^0 > \vee < E > = < S^0 >$
4. $< pt > \wedge < E > = < pt >$
5. $< pt > \vee < E > = < E >$

Proposition 16. *(7.2.2) The localization functors L_E and L_F are the same if and only if $< E >=< F >$. If $< E >\leq< F >$, then $L_E L_F = L_E$ and there is a natural transformation $L_F \to L_E$.*

Let $S^0_{\mathbf{Q}}$ denote the rational sphere spectrum (the initial spectrum whose homotopy groups are vector spaces over $\mathbb{Q}$ and $f : S^0 \to S^0_{\mathbf{Q}}$ induces equivalence on rational homotopy groups). Let $S^0_{(p)}$ denote the p-local sphere spectrum and $S^0/(p)$ denote the mod p Moore spectrum (cofiber of $S^0 \xrightarrow{p} S^0$) We have the following result.

Proposition 17. *(7.2.5)*

$$
\begin{aligned}
< S^0_{(p)} > &=< S^0_{\mathbf{Q}} > \vee < S^0/(p) > \\
< pt > &=< S^0_{\mathbf{Q}} > \wedge < S^0/(p) > \\
< pt > &=< S^0/(q) > \wedge < S^0/(p) >, p \neq q \\
< S^0 > &=< S^0_{\mathbf{Q}} > \vee \bigvee_p < S^0/(p) >
\end{aligned}
$$

Definition 18. (Smashing) A spectrum E is called smashing if

$$< E >=< L_E S^0 > .$$

Proposition 19. *(Smashing) If E is smashing, then*

1. *$X \xrightarrow{1\wedge\eta} X \wedge L_E S^0$ is an E_*-localization.*
2. *Every direct limit of E_*-local spectra is E_*-local.*
3. *L_E commutes with direct limits.*

Proposition 20. *If*

$$X \to X \xrightarrow{f} Y \to \Sigma W$$

is a cofiber sequence, then

$$< W >\leq< X > \vee < Y >$$

with equality holds when f is smash nilpotent. For a self map (not necessarily v_n) f, let C_f denote its cofiber and $\hat{X}$ denote the telescope obtained by iterating f. Then we have

$$< X >=< \hat{X} > \vee < C_f >$$

$$< \hat{X} > \wedge < C_f >=< pt > .$$

Theorem 21. *(7.2.7 Class invariance) Let X and Y be p-local finite CW-complexes of types m, n. Then $< X >=< Y >$ if and only if $m = n$, and $< X ><< Y >$ if and only if $m > n$.*

Proof. We can form C_X and C_Y, the smallest thick subcategory of $FH_{(p)}$ containing X and Y. Since everything in a thick subcategory is built up from X using cofibration and retraction, for any $X' \in C_X$ we have

$$< X' ><< X >$$

using the last proposition. We can then show that $C_X \in F_m$ and $C_X \notin F_{m+1}$ because X is of type m, which means $C_X = F_m$ and $C_Y = F_n$. When $m = n$, then $C_X = C_Y$ so $< X >=< Y >$. □

3 MU

Definition 22. $(MU, BP, E(n), K(n), H/(p))$ Recall that

$$< MU >= \bigvee_p < MU_{(p)} >= \bigvee_p < BP >$$

is a wedge sum of BP with various primes p.

Theorem 23. *(7.4.1 Brown-Comenetz dual) Let Y be a spectrum with finite homotopy groups. Then there is a spectrum cY, called the Brown Comenetz dual of Y such that $ccY = Y$ and for any spectrum X,*

$$[X, cY]_{-i} = Hom(\pi_i(X \wedge Y), \mathbb{R}/\mathbb{Z}).$$

In particular, we have

$$\pi_{-i}(Y) = Hom(\pi_i(Y), \mathbb{R}/\mathbb{Z})$$

and $cH/(p) = H/(p)$.

It follows that if $[X, cY]_* = 0$, then we $X \wedge Y$ is contractable. Similarly, if $[X, Y]_* = 0$, then $X \wedge cY$ is contractable. Let X be MU and Y be a finite spectrum with trivial rational homology, using the Adams spectral sequence one can show that $[X, Y]_* = 0$ so $MU_*(cY) = 0$. Therefore, there exists a non-contractable spectrum with $MU_*(cY) = 0$, so we know that $< MU >< < S^0 >$. Moreover, we have the following result.

Theorem 24. *(7.4.2 $< MU >< < S^0 >$) There is a spectrum $X(n)$ for $1 \leq n \leq \infty$ with $X(1) = S^0$ and $X(\infty) = MU$ such that $< X(n) > \geq < X(n+1) >$ for each n with*

$$< X(p^k - 1)_{(p)} > > < X(p^k)_{(p)} >$$

for each prime p and each $k \geq 0$.

The Bott periodicity gives us a homotopy equivalence $\Omega SU \to BU$ and by composing this map with the inclusion of $SU(n)$ into SU we get

$$\Omega SU(n) \to BU$$

whose associate Thom spectrum is $X(n)$. We have

$$H_* X(n) = \mathbb{Z}[b_1, ..., b_{n-1}]$$

where $|b_i| = 2i$ and the generators map to generators of the same name of H_*MU.

4 $E(n)$-Localization

Definition 25. (7.5.1) We write $L_n X$ for $L_{E(n)}X$ and let $C_n X$ be the fiber of the localization $X \to L_{E_n} X$.

Theorem 26. *(7.5.2 Localization theorem) For any spectrum Y, we have*

$$BP \wedge L_n Y = Y \wedge L_n BP.$$

Moreover, if $v_{n-1}^{-1} BP_*(Y) = 0$, we have $BP \wedge L_n Y = Y \wedge v_n^{-1} BP$, i. e. $BP_*(L_n Y) = v_n^{-1} BP_*(Y)$.

Definition 27. (7.5.3 Chromatic tower) The chromatic tower for a p-local spectrum X is the inverse system

$$L_0 X \leftarrow L_1 X \leftarrow L_2 X \leftarrow ...$$

We know for a fact that

$$< E(n) > = \bigvee_{i=0}^{n} < K(i) >$$

so

$$< E(n) > = < E(n-1) > \vee < K(n) > \leq < E(n-1) >$$

and so we have

$$L_{n-1} \to L_n$$

Conjecture 28. *(7.5.5 Telescope) Let X be a p-local finite CW-complex of type n with a v_n map f. Let $\hat{X}$ be the telescope, and we know $K(n)_* f$ is an isomorphism and $K(i)_* f = 0$ for $i < n$. Therefore, we know $E(n)_* f$ is an equivalence so $X \to L_n X$ factors through the telescope $\hat{X}$, so we have a map*

$$\lambda : \hat{X} \to L_n X.$$

Moreover, we get

$$BP_*(L_n X) = v_n^{-1} BP_*(X)$$

and λ is a BP-equivalence by the localization theorem.