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Chromatic Talk

Recall:

- $L :=$ Lazard ring, univ fgl G
- $\Gamma := \{\gamma = x + b_1 x^2 + b_2 x^3 + \dots\} \subseteq \mathbb{Z}[[x]]$

Group under composition

- $\Gamma \circlearrowright L$ by sending $G(x,y) \mapsto \gamma^{-1}(G(\gamma(x), \gamma(y)))$

$$\Longleftrightarrow \gamma_* : L \to L$$

- $C\Gamma \subset \mathrm{Mod}_L$ w/ compatible Γ actions

ie $\gamma(\lambda m) = \gamma_*(\lambda)\, \gamma(m)$ $\forall \gamma \in \Gamma$, $\lambda \in L$, $m \in M$

- $FH := h\, \mathrm{Spectra}^{fin}$

- $$\begin{tikzcd} & C\Gamma \arrow[d] \\ FH \arrow[ur, dashed, "MU_*"] \arrow[r, "MU_*"'] & Ab^{\mathbb{Z}} \end{tikzcd}$$

② v_n = coeff of x^{p^n} in $\underbrace{F(x, F(x, \cdots))}_{p \text{ times}}$ mod p

- $I_{p,n} = \langle p, v_1, \ldots, v_{n-1} \rangle \subset L$

<u>Thm</u> [Invariant Prime Ideal]

① The only prime ideals of L invariant under $\Gamma \circlearrowright L$ are $I_{p,n}$

② $\left(\frac{L}{I_{p,n}}\right)^{\Gamma} \simeq \frac{\mathbb{Z}}{p}[v_n]$

<u>Thm</u> [Landweber filtration]

$\forall M \in C\Gamma$, $\exists$ a finite filtration

$$0 = F_0 M \subseteq F_1 M \subseteq \cdots \subseteq F_k M = M$$

s.t. $\frac{F_i M}{F_{i-1} M} \simeq \frac{L}{I_{p,n}}[k]$ (for some p, n, k)

<u>Def</u> ① A full subcat $C \subseteq C\Gamma$ is <u>thick</u> if $\forall$ SES

$$0 \to M' \to M \to M'' \to 0$$

$$M', M'' \in C\Gamma \iff M \in C\Gamma$$

② A full subcat $F \subseteq FH$ is <u>thick</u> if $\forall$ cofib seq

$$X \to Y \to C_f$$

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we have $\frac{2}{3} \in F \Rightarrow 3^{rd} \in F$
(e $X \vee Y \in F \Rightarrow X, Y \in F$)

3 $M \in C\Gamma$ is <u>p-local</u> if it admits a module structure for $L_{(p)}$
$\Leftrightarrow$ only has p-torsion

$C\Gamma_{(p)} \subseteq C\Gamma$ is the full subcat of p-local L-modules

Similarly, have $FH_{(p)} \subseteq FH$
($\pi_* X$ " p-local)

<u>Thm</u>: Let $C \subseteq C\Gamma_{(p)} \subseteq C\Gamma$ be thick
then either
a) $C = C\Gamma_{(p)}$
b) $C = *$
c) $C = C_{p,n} := \{M \in C\Gamma_{(p)} \mid v_{n-1}^{-1} M = 0\}$

($\Leftrightarrow \forall m \in M \; \exists k$ st $v_{n-1}^{k} m = 0$)

<u>Cor</u>: p-locally, we have a filtration of thick subcats

$C\Gamma_{(p)} = C_{p,0} \supseteq C_{p,1} \supseteq \cdots \supseteq *$

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Goal: Get similar filtration for $FH(p)$

Plan: Take a detour through

Landweber exact functor theorem

$\rightsquigarrow$

Morava K-theory

$\rightsquigarrow$

Spanier-Whitehead duality

$\rightsquigarrow$

end goal

Fix p →

Q: For what BP_*-module M is

$$X \longmapsto BP_*(X) \otimes_{BP_*} M$$

a homology theory?

Issue: Won't preserve exact seq in general.
Implied by flatness, but that is rare

Fix: We only need $\mathrm{Tor}_1^{BP_*}(M, BP_*(X)) = 0 \quad \forall X$

$$\overset{LFT}{\Longleftrightarrow} \mathrm{Tor}_1^{BP_*}(M, BP_*/I_n) = 0 \quad \forall n$$

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(BP-version)

Thm [Landweber Exact Functor]

The functor $X \mapsto BP_*(X) \otimes_{BP_*} M$ is exact

$\iff \forall n$, mult by v_n is monic in $M \otimes_{BP_*} BP_*/I_n$

$\iff \exists$ BP_*-module spectrum E s.t. $E_* = M$

Eg: $E(n)$, Morava E-theory, where

$$M = E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n, v_n^{-1}]$$

MU version exists too

Issue: Want Morava K-theory $K(n)$ s.t.

$$K(n)_* = \frac{\mathbb{Z}}{p}[v_n, v_n^{-1}]$$

This is not Landweber exact!

Fixes ① Get SS

$$E_2 = \mathrm{Tor}^{MU_*}(MU_*(X), K(n)_*) \Rightarrow K(n)_*(X)$$

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2 a) Construct by hand $P(n)$ s.t.

$$P(n)_* = BP_*/I_n$$

(also fails Landweber exactness)

b) Prove $P(n)$ verson of LEFT

c) Show $K(n)_*$ is $P(n)$ Landweber exact

$$\Rightarrow K(n) : X \mapsto P(n)_*(X) \otimes_{P(n)_*} K(n)_*$$

Facts: 1 $K(n)$ has a Künneth iso

$$K(n)_*(X \times Y) \simeq K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y)$$

2 $K(n)$ is a ring spectrum

3 $K(0)_*(X) \simeq H_*(X; \mathbb{Q})$

$$K(1)_*(X) \simeq \bigvee_{p-1} \frac{KU_*(X)}{p}$$

4 $K(n)_*(X) = 0$

$\Rightarrow K(n-1)_*(X) = 0$

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Def/Thm [Spanier-Whitehead] $\forall X \in FH$ $\exists!\, DX \in FH$ s.t.

$\forall$ spectra Y

1. $[X, Y]_* \simeq (DX \wedge Y)_* \simeq [DX, DY]_*$

2. $\mathrm{Hom}_{K(n)_*}(K(n)_*(X), K(n)_*(Y)) \simeq K(n)_*(DX \wedge Y)$

3. $DDX \simeq X$

4. $D(X \wedge Y) \simeq DX \wedge DY$ (Y finite)

5. D is contravariant

Using this + some big hammers in Ch 7,

Thm [Nilpotence] Let $W, X, Y \in FH_{(p)}$, and $f: X \to Y$. Then

$$K(n)_*(W \wedge f) = 0 \quad \forall n$$

$$\Rightarrow W \wedge f^{\wedge k} = 0 \quad \text{for } k \gg 0$$

"All the $K(n)$'s together detect Nilpotence"

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Def: $F_{p,n} = \{ X \in FH_{(p)} \mid K(n-1)_*(X) = 0 \}$

$(\Rightarrow K(i)_*(X) = 0 \quad \forall i \leq n)$

Thm: Let $F \subseteq FH_{(p)} \subseteq FH$ be thick. Either

a) $F = FH_{(p)}$
b) $F = *$
c) $F = F_{p,n}$ for some n