Lecture Notes on MU Theory

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February 14, 2023

Before we start describing the construction and properies of the spectrum MU, we will motivate why on Earth we should care about this. We will soon see that MU possesses rich algebraic structure that imbues it with the ability to detect important algebraic properties *of other spectra.*

Explicitly, over the course of this reading group, one of the results we will discuss is the following

Theorem 0.1 – Nilpotence Theorem - Spectra Version For any ring spectrum R, $x \in \pi_*(R)$ is *nilpotent* if it is in the kernel of the Hurewicz/Boardman map $\pi_*(R) \to MU_*(R)$.

1 Geometry of MU Theory

This description of MU theory develops off of the theory of complex vector bundles, classifying spaces, and characteristic classes. I assume that the following concepts are already familiar to the reader:

- Complex Vector Bundles $p : E \to B$
- Sum and Product operators on complex vector bundles
- Grassmannians $G^F_{n,k}$ and Universal Bundles $\gamma_{n,k}$
- Classifying spaces BU, BO, etc,

Theorem 1.1 $H^*(BU(n), \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \ldots, c_n]$ with $|c_i| = 2i$.
Similarly, $H^*(BO(n), \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[w_1, \ldots, w_n]$ with $|w_i| = i$.

The generators of $H^*(BU(n))$ are called *Chern Classes* and the generators of $H^*(BO(n))$ are *Steifel-Whitney classes.*

1.1 Thom Spectra

If the base space B of a complex vector bundle is *paracompact*, we can endow it with a *Hermitian Metric*, which allows us to define auxilliary bundles.

Definition 1.1 – Disk and Sphere Bundles Given a complex vector bundle $p : E \to B$ with a Hermitian metric $\langle -, - \rangle : E \times E \to \mathbb{C}$, the *Disk Bundle* $D(p)$ associated to the complex vector bundle is a bundle whose fiber over $b \in B$ is $\{e \in E : \langle e, e \rangle \leq 1\}$. Similarly, the *Sphere Bundle* $S(p)$ is a bundle whose fiber over $b \in B$ is $\{e \in E : \langle e, e \rangle = 1\}$.

With these definitions, we can construct the *Thom Space* associated to the bundle:

Definition 1.2 – Thom Space With $p : E \to B$ and $\langle -, - \rangle : E \times E \to \mathbb{C}$ as above, the *Thom Space* $T(p)$ of the bundle is the total space of the quotient bundle $D(p)/S(p)$ (with the quotient induced by the obvious inclusion).

Definition 1.3 – $MU(n)$ With G_n the colimit over the $G_{n,k}$'s, and γ_n defined similarly, we define $MU(n) = T(\gamma_n)$ in the complex case, and $MO(n) = T(\gamma_n)$ in the real case.

Since the Thom Space of a bundle is, in a sense, built up from quotients of $D^{2n}/S^{2n-1} \approx S^{2n}$, there is a cohomology class in $T(p)$ that acts as the generator of these spheres fiberwise:

Definition 1.4 – Thom Class Associated to any complex vector bundle $p : E \to B$ of rank n, there is a class $u \in H^{2n}(T(p), \mathbb{Z})$ called the *Thom Class* so that its restriction to the fibers S^{2n} is a generator.

Most importantly, these $MU(n)$ spaces 'glue together' and form a spectrum called the *Thom Spectrum*

Definition 1.5 – MU **- The Thom Spectrum** Defining $MU_{2n} = MU(n)$ and $MU_{2n+1} = \Sigma MU(n)$, we obtain a spectrum with $\Sigma MU_{2n} \to MU_{2n+1}$ as the obvious map, and the map $\Sigma MU_{2n+1} \to MU_{2(n+1)}$ defined by the following proposition.

Proposition 1.1 $T(\xi_1 \oplus \xi_2) \cong T(\xi_1) \wedge T(\xi_2)$. In particular, If a vector bundle ξ is isomorphic to $\xi' \oplus \epsilon$ with ϵ the trivial complex line bundle, then $T(\xi) \approx \Sigma^2 T(\xi')$

Additionally, we have a map $j : BU(n) \to BU(n+1)$ which classifies $\xi \mapsto \xi \oplus \epsilon$. From this, $\Sigma MU_{2n+1} = \Sigma^2 MU_{2n} \approx T(\gamma_{2n} \oplus \epsilon) \xrightarrow{T(j)} T(BU(n+1)) = MU(n+1) = MU_{2(n+1)}$ where $T(j)$ denotes the pullback of j on the Thom spaces.

We can also define a multiplicative structure on the Thom Spectrum via the Thom-ification of the map classifying direct sums of vector bundles: $m : BU(n) \times BU(m) \to BU(n+m)$, so $T(m) : MU(n) \times MU(m) \to MU(n+m)$ gives MU the structure of a ring spectrum.

Thus, we can speak of the homology and cohomology theory that MU describes, which are complex bordism and cobordism respectively.

While the spectrum has a lovely geometric interpretation in this manner, it's algebraic structure is much more interesting for our purposes.

1.2 Complex Orientability, Universality, and Formal Group Laws

MU^* is a *complex oriented cohomology theory* i.e.

Definition 1.6 – Complex Orientability, Complex Orientation A cohomology theory E^* is called complex orientable if the map $i^* : \tilde{E}^2(\mathbb{C}P^\infty) \to \tilde{E}^2(S^2) \cong \tilde{E}^0(S^0) \cong \pi_*(E)$ is surjective. A complex orientation on such a cohomology theory is a choice of element in the preimage of $1 \in \pi_*(E)$, i.e. an element $x^E \in \tilde{E}^2(\mathbb{C}P^\infty)$ so that $i^*(x^E) = 1$.

Since $\mathbb{C}P^\infty \approx MU(1)$ and $\mathbb{S} \approx MU(0)$ the chosen homeomorphism in the former case gives an element $x^{MU} \in MU^2(\mathbb{C}P^\infty)$ and the map $S^2 = \Sigma^2\mathbb{S} = \Sigma^2 MU(0) \to MU(1) = \mathbb{C}P^\infty$ corresponds to the inclusion $i^* : \tilde{MU}^2(\mathbb{C}P^\infty) \to \tilde{MU}^2(S^2)$ so $i^*(x^{MU})$ must be the class of the unit $\eta : \mathbb{S} \to MU$ which is 1. Therefore x^{MU} defines a complex orientation for MU^*. Moreover, $MU^*(\mathbb{C}P^\infty) \cong MU^*[[x^{MU}]]$.

We will see that this complex orientation structure gives rise to a natural formal group law on $\pi_*(MU)$. In this direction, note that there is a product structure on $\mathbb{C}P^\infty$ $m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ which classifies the tensor product of line bundles. On the cohomology level, this induces a map $m^* : MU^*[[x]] = MU^*(\mathbb{C}P^\infty) \to MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong MU^*[[x \otimes 1, 1 \otimes x]]$.

Proposition 1.2 $m^*(x) = F(x \otimes 1, 1 \otimes x)$ defines a formal group law over $\pi_*(MU)$

From the previous lecture, we know that any formal group law over a ring R is classified by a map $\theta : L \to \pi_*(MU)$. MU is very special in that

Theorem 1.2 – Quillen's Theorem The map θ above is an isomorphism of rings and an isomorphism between the universal F.G.L. and the one defined above on $\pi_*(MU)$.

As an aside, assuming the nilpotence theorem, since every (positive degree) element in $\pi_*(\mathbb{S})$ is torsion, and $MU_*(\mathbb{S}) = \pi_*(MU) \cong L$ is the Lazard ring (which is torsion free) we immediately deduce

Theorem 1.3 – Nishida's Nilpotence Theorem Every positive degree element in $\pi_*(\mathbb{S})$ is nilpotent.

Additionally, there is a very strong connection between complex orientations and formal group laws (mimicing the construction of the F.G.L. in Prop 1.2) in which MU acts as the 'spectra analogue' of L.

Theorem 1.4 – Universality of MU For any complex oriented multiplicative cohomology theory E, choices of complex orientations x^E are in 1-to-1 correspondence with homotopy classes of maps $f : MU \to E$ such that $f_*(x^{MU}) = x^E$ and so that for the F.G.L. $\mu^E(x^E \otimes 1, 1 \otimes x^E) = m^*(x^E)$, $f_*(\mu^{MU}) = \mu^E$.

2 Algebra of MU Theory

Our goal this section will be to study the structure of $MU_*(MU)$, which we will motivate by the corresponding story in a more familiar homology theory.

2.1 Steenrod Operations, Eilenberg-MacLane Spectra, and Hopf Algebras

Let E denote the mod p Eilenberg-MacLane spectra, so that $E^*(X) = H^*(X; \mathbb{Z}/_{(p)})$. This cohomology theory is also equipped with a collection of 'endomorphisms' called the *Steenrod operations* which form an algebra A over which the cohomology groups have a natural module structure.

I.e. we have a map

$$A \otimes E^*(X) \to E^*(X)$$

Dually, this describes a *comodule structure*

$$E_*(X) \xrightarrow{\psi} A^\vee \otimes E_*(X) \tag{1}$$

Interestingly, $A \cong E^*(E)$, so $A^\vee \cong E_*(E)$.

Furthermore, we can rewrite Equation 1 to the following:

$$E_*(X) \xrightarrow{\psi} \pi_*(E \wedge E \wedge X) \tag{2}$$

Which is induced by a map on the level of spectra:

$$E \wedge X = E \wedge \mathbb{S} \wedge X \xrightarrow{E \wedge \eta \wedge X} E \wedge E \wedge X \tag{3}$$

And since E is a flat ring spectrum (which MU also is) the map $E_*(E) \otimes_{E_*} E_*(X) \to \pi_*(E \wedge E \wedge X)$ is an isomorphism.

Algebraically, the dual Steenrod Algebra has the structure of a Hopf Algebra, a structure which we will present in two different manners

Definition 2.1 – Hopf Algebra - Explicit An R-algebra A, equipped with product $A \otimes A \xrightarrow{\mu} A$ and unit $R \xrightarrow{\eta} A$ is a Hopf algebra, if it is equipped with other maps called the coproduct $A \xrightarrow{\Delta} A \otimes A$, augmentation $A \xrightarrow{\epsilon} R$, and conjugation $A \xrightarrow{c} A$ such that the group structure (from the product, conjugation, and unit) and cogroup structure (from the coproduct, conjugation, and augmentation) are 'compatible'.

Definition 2.2 – Hopf Algebra - Categorical A Hopf Algebra A over R is a cogroup object in the category of (graded commutative) R algebras with unit. I.e. for all other algebras C, there is a natural group structure on $\mathrm{Hom}(A, C)$.

For any ring spectrum E, not just the Eilenberg-MacLane spectrum in consideration, the product, coproduct, unit, and conjugation maps are all manifestations of maps on the topological objects (after identifying A_* with $\pi_*(E \wedge E)$, $\mathbb{Z}/_{(p)}$ with $\pi_*(E)$, and $A_* \otimes A_*$ with $\pi_*(E \wedge E \wedge E)$).

Unfortunately, for general ring spectra the unit map $\pi_*(E) \xrightarrow{\epsilon} \pi_*(E \wedge E)$ does not have a single topological counterpart, but rather two — a left and right unit. Coincidentally, these induce the same maps on E homology in the case of the Eilenberg-MacLane spectra, but we need to further generalize for other spectra. Such a generalization will lead us to *Hopf Algebroids*.

2.2 $MU_*(MU)$ and Hopf Algebroids

Let us examine the problem more explicitly. Central to the problem is the fact that we have two spectra-level maps

$$\eta_R : E = \mathbb{S} \wedge E \xrightarrow{\eta \wedge E} E \wedge E \tag{4}$$

$$\eta_L : E = E \wedge \mathbb{S} \xrightarrow{E \wedge \eta} E \wedge E \tag{5}$$

So that $E^*(E)$ becomes a $\pi_*(E)$ module in two different ways, which we call the right and left module structures. It turns out that these structures are how $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ is structured as a bimodule, i.e. $\pi_*(E)$ acts as a unit via the left unit on the left term, and the right unit on the right term. To motivate the shift to Hopf Algebroids, notice that the tensor product diagram after composing with a $\mathrm{Hom}(-, C)$ is

$$\begin{array}{ccc} \mathrm{Hom}(E_*(E) \otimes_{\pi_*(E)} E_*(E), C) & \longrightarrow & \mathrm{Hom}(E_*(E), C) \\ \downarrow & & \downarrow \eta_R^* \\ \mathrm{Hom}(E_*(E), C) & \xrightarrow{\eta_L^*} & \mathrm{Hom}(\pi_*(E), C) \end{array} \tag{6}$$

Is 'like' the diagram describing the composition of maps in a category. In fact, this is the manner by which we will generalize Hopf Algebras to the case where there are multiple unit maps, so that the cogroup structure is instead a cogroupoid structure.

Definition 2.3 – Hopf Algebroid A Hopf Algebroid over K is a pair (S, Σ) of graded commutative K-Algebras with unit, with the additional structure of a cogroupoid object in that category, i.e. for any algebra C, $(\mathrm{Hom}(S, C), \mathrm{Hom}(\Sigma, C))$ has the structure of a groupoid, with the former being the set of objects and the latter being the set of morphisms. Explicitly, the structure is

$$\Sigma \xrightarrow{\Delta} \Sigma \otimes_S \Sigma \tag{7}$$

$$\Sigma \xrightarrow{c} \Sigma \tag{8}$$

$$\Sigma \xrightarrow{\epsilon} S \tag{9}$$

$$S \xrightarrow{\eta_R} \Sigma \tag{10}$$

$$S \xrightarrow{\eta_L} \Sigma \tag{11}$$

which, after applying $\mathrm{Hom}(-, C)$ induce composition, inverses, identity morphisms, target, and source respectively. $\Sigma \otimes_S \Sigma$ is defined similarly to Diagram 6.

In light of Diagram 6 we have the following theorem:

Theorem 2.1 If E is a flat, homotopy commutative ring spectrum, then $(\pi_*(E), E_*(E))$ is a Hopf Algebroid over $\mathbb{Z}$. If E is p-local, then it is a Hopf Algebroid over $\mathbb{Z}_{(p)}$.

Now, MU satisfies all of the conditions of the above theorem, so, unwinding the definitions, $(\pi_*(MU), MU_*(MU))$ forms a Hopf Algebroid. Therefore, for every graded $\mathbb{Z}$ algebra C, $\mathrm{Hom}(\pi_*(MU), C)$ forms the objects in a category whose morphisms are $\mathrm{Hom}(MU_*(MU), C)$.

Remark 2.1 We know that $\pi_*(MU) \cong L$ the Lazard ring, which classifies formal group laws. Thus $\text{Hom}(\pi_*(MU), C)$ is the set of all formal group laws over C! We will see soon that $MU_*(MU) \cong LB$, the classifying ring for strict isomorphisms of formal group laws as well as that the manner in which it acts on $\pi_*(MU)$ is the same as the case for the Lazard Ring. Thus the Hopf Algebroid associated to MU exactly classifies formal group laws and the strict isomorphisms between them (over a given ring C).

With this formulation, we can now form left comodules over Hopf Algebroids

Definition 2.4 – Left Comodule A Left Comodules over a Hopf Algebroid (S, Σ) is an S-module M and a left S-linear map $M \xrightarrow{\psi} \Sigma \otimes_S M$ that is counital and coassocaitive.

Right Comodules can be defined similarly, and comodule algebras are (left/right) comodules such that M is an S-algebra and that ψ is a algebra homomorphism.

Again, these structures arise naturally when we study spectra and their generalized homology:

Proposition 2.1 For E a flat, homotopy commutative ring spectrum, $E_*(X)$ is naturally a left comodule over $(\pi_*(E), E_*(E))$, with the structure map topologically induced by $E \wedge X = E \wedge \mathbb{S} \wedge X \xrightarrow{E \wedge \eta \wedge E} E \wedge E \wedge X$

2.3 The Hopf Algebroid Structure of $MU_*(MU)$

This section is devoted to fully examining the connection we hinted at in Remark 2.1. In order to provide the names to some elements later on in the section we once again discuss the homotopy groups of MU

Theorem 2.2 – Milnor-Novikov

- $\pi_*(MU) \cong \mathbb{Z}[x_1, \ldots]$ where $|x_i| = 2i$
- The generators x_i can be chosen so that their image under the Hurewicz homomorphism $h : \pi_*(MU) \to H_*(MU) \cong \mathbb{Z}[b_1, \ldots]$ (where $|b_i| = 2i$) is

$$h(x_i) = \begin{cases} pb_i + \textit{decomposables}, & \text{if } i = p^k - 1 \\ b_i + \textit{decomposables}, & \text{otherwise} \end{cases} \tag{12}$$

We also label other special elements $m_n \in \pi_{2n}(MU) \otimes \mathbb{Q}$ that come from classes of certain cobordism classes of complex projective spaces.

The following theorem is what will connect the Hopf Algebroid structure of $(\pi_*(MU), MU_*(MU))$ to that of the Lazard ring and the classification of formal group laws.

Theorem 2.3 – Landweber-Novikov

- $MU_*(MU) \cong MU_*[b_1, \ldots]$ with $|b_i| = 2i$.
- The coproduct on $MU_*(MU)$ is given by $\sum_{i\geq 0} \Delta(b_i) = \sum_{i\geq 0}(b_i \otimes (\sum_{j\geq 0} b_j)^{i+1})$ (where we interpret $b_0 = 1$).
- The left unit $\eta_L : MU_* \to MU_*(MU)$ is just the inclusion into degree 0.
- The right unit η_R is given by $\sum_{i\geq 0} \eta_R(m_i) = \sum_{i\geq 0}(m_i(\sum_{j\geq 0} c(b_j))^{i+1})$
- Conjugation is given by $c(m_n) = \eta_R(m_n)$ and $\sum_{i\geq 0}(c(b_i)(\sum_{j\geq 0} b_j)^{i+1}) = 0$.

Since the coproduct on $MU_*(MU)$ described above functions exactly like one on $B = \mathbb{Z}[b_1, \ldots]$, and is moreover determined by such a structure, as well as the right unit map, the algebroid structure is of a special type called a *split Hopf algebroid*. Furthermore, the right unit map $MU_* \to MU_* \otimes B$ is a comodule structure map, so that MU_* becomes a right B-comodule.

Definition 2.5 – Split Hopf Algebroid Generally, a Hopf Algebroid (S, Σ) is *split* if there if a Hopf Algebra B such that $\Sigma = S \otimes B$ and so that the right unit $\eta_R : S \to \Sigma = S \otimes B$ gives S the structure of a right B-comodule.

The motivation for the term 'split' is because for such Hopf Algebroids,

$$\mathrm{Hom}(\Sigma, C) = \mathrm{Hom}(S \otimes B, C) = \mathrm{Hom}(S, C) \otimes \mathrm{Hom}(B, C) \tag{13}$$

and

$$\begin{array}{c} S \xrightarrow{\eta_R} S \otimes B \\ \Downarrow \mathrm{Hom}(-, C) \\ \mathrm{Hom}(S, C) \otimes \mathrm{Hom}(B, C) \xrightarrow{\mathrm{Hom}(\eta_R, C)} \mathrm{Hom}(S, C) \end{array} \tag{14}$$

So the morphism group splits, and the group $\mathrm{Hom}(B, C)$ has a right action on $\mathrm{Hom}(S, C)$.

From Dinglong's talk, we know that the group of isomorphisms between formal group laws over a ring R is given by $\Gamma_R = \{f \in R[[x]] : f = x + \sum_{i\geq 1} b_i x^{i+1}\}$. We will now construct an isomorphism $\mathrm{Hom}(B, R) \xrightarrow{\cong} \Gamma_R$

Theorem 2.4 For $g \in \mathrm{Hom}(B, R)$, let $f(x) \in \Gamma_R$ be $f(x) = \sum_{i\geq 0} f(b_i)x^{i+1}$. This gives a map $\mathrm{Hom}(B, R) \xrightarrow{F} \Gamma_R$ which is an isomorphism.

Proof. Since B is freely generated, the map is obviously a bijection; the only nontrivial element to prove is that it is a homomorphism. However, composing $g_0(x) = \sum_{i\geq 1} b_i x^{i+1}$ with $g_1(x) = \sum_{i'\geq 1} b'_i x^{i+1}$ is exactly the formula for the coproduct on B, and so F is a homomorphism, thus a isomorphism. ∎

To connect $MU_*(MU)$ with the isomorphisms between F.G.L.'s explicitly, we will use the logarithm of a formal group law introduced in the previous talk and two important properties of it:

Theorem 2.5 $\log_F(F(x,y)) = \log_F(x) + \log_F(y)$ for a formal group law F over a torsion free ring R.

and

Theorem 2.6 – Mischenko $\log_G(x,y) = \sum_{i\geq 0} m_i x^{i+1}$, where $m_i \in \pi_{2i}(MU)$ are the elements introduced at the start of the section, and G is the universal formal group law.

We prove one more auxiliary proposition, motivated by the desire to use the logarithm to help facilitate a connection between a formal group law F and its tranform F^f under $f \in \Gamma_R$.

Proposition 2.2 $\log_{F^f}(x) = \log_F(f^{-1}(x))$

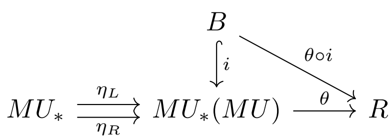
Proof.
The definition of how f acts on F can be altered to give that $f^{-1}(F^f(x,y)) = F(f^{-1}(x), f^{-1}(y))$. We then apply $\log_F$ to get

$$\begin{aligned} \log_F(f^{-1}(F^f(x,y))) &= \log_F(F(f^{-1}(x), f^{-1}(y))) \\ &= \log_F(f^{-1}(x)) + \log_F(f^{-1}(y)) \end{aligned} \tag{15}$$

while also noting that $\log_{F^f}(F^f(x,y)) = \log_{F^f}(x) + \log_{F^f}(y)$. Ravenel notes that you can compare the two degree by degree, and proceed via induction to show the result of the theorem. I haven't been able to tease out the details, but I'm fairly certain the 'degree' comparison Ravenel mentions is the comparison of Taylor Series.

□

We now explicitly connect $(MU_*, MU_*(MU))$ to the Hopf algebroid of F.G.L.'s and the isomorphisms between them. We do this with some given $\theta \in \mathrm{Hom}(MU_*(MU), R)$



(16)

Since $\theta \circ \eta_L$ and $\theta \circ \eta_R$ are both maps $MU_* \to R$ they classify formal group laws F, F' over R respectively. The composite $\theta \circ i : B \to R$ classifies some isomorphism between formal group laws $f \in R[[x]]$. By the universality of MU_* and Theorem 2.6, we can compute the logarithms of F and F' as $\log_F(x) = \sum_{i\geq 0} \theta(\eta_L(m_i))x^{i+1} = \sum_{i\geq 0} \theta(m_i)x^{i+1}$ and $\log_{F'}(x) = \sum_{i\geq 0} \theta(\eta_R(m_i))x^{i+1}$. We would like to be able to show $\log_{F'}(x) = \log_{F^f}(x) = \log_F(f^{-1}(x))$.

$$\log_F(f^{-1}(x)) = \tag{17}$$

$$\sum_{i\geq 0} \theta(m_i)(f^{-1}(x))^{i+1} = \tag{18}$$

$$\sum_{i\geq 0} \theta(m_i)(\sum_{j\geq 0} \theta(c(b_j))x^{j+1})^{i+1} = \tag{19}$$

$$\theta(\sum_{i\geq 0} m_i(\sum_{j\geq 0} c(b_j)x^{j+1})^{i+1}) = \tag{20}$$

$$\sum_{i\geq 0} \theta(\eta_R(m_i))x^{i+1} = \log_{F'}(x) \tag{21}$$

So $F^f = F'$ and thus we have shown that $(\pi_*(MU), MU_*(MU))$ is 'isomorphic' to (L, LB) of the previous lecture.

2.4 Further Discussion of the Algebraic Structure On $MU_*(X)$

Before we dissect the structure of $MU_*(MU)$ we perform a few algebraic constructions.

Firstly, we consider the following group $\Gamma = \{\gamma \in \mathbb{Z}[[x]] : \gamma = x + \sum_{i=1}^{\infty} b_i x^{i+1}\}$ with the group structure induced by composition of power series. This group acts on the Lazard ring (and therefore $\pi_*(MU)$) in the following manner. Given the universal formal group law $G(x, y) \in L[[x, y]]$ and $\gamma \in \Gamma$, we can construct a new formal group law $\gamma G = \gamma^{-1}(G(\gamma(x), \gamma(y)))$ which, by the universality of the Lazard ring, is induced by a self map $f_\gamma : L \to L$. As γ is an invertible power series, this self map must be an automorphism, which is exactly the structure of an action of Γ on L.

We encode this structure for L modules as follows

> **Definition 2.6** – $C\Gamma$ Let $C\Gamma$ be a category whose objects are L-modules M together with an action by Γ that is compatible with the L-module structure, i.e. $\gamma(\ell m) = \gamma(\ell)\gamma(m)$ for all $\gamma \in \Gamma, \ell \in L, m \in M$.

Importantly, every MU-homology group has both the structure of a $L \cong \pi_*(MU)$-module, as well as an action by Γ which has been developed in the preceding section. For now, the important point is that MU_* thus becomes a functor into the very well behaved category $C\Gamma$, with its structure and behavior being responsible for the major theorems we will prove over this semester.

Additionally, we will let the elements $v_n \in L$ be the coefficient of x^{p^n} in the universal formal group law $G(x, y)$ over L. These v_n's are related to those from Morava K-Theory which we will see in later lectures, and which describe periodic 'families' in the stable homotopy groups of spheres. These elements have degree $2p^n - 2$ and are *indecomposable*, which means they could serve as polynomial generators in their dimension.

These v_n's encode a lot of structure about modules in $C\Gamma$ via the following:

Theorem 2.7 – Invariant Prime Ideal Theorem The only prime ideals in L which are invariant under the action of Γ are ideals of the form $I_{p,n} = \langle p, v_1, v_2, \ldots, v_n \rangle$ for $0 \leq n \leq \infty$. In $L/I_{p,n}$, the only subgroup that is fixed by Γ is $\mathbb{Z}/(p)[v_n]$.

Thus the manner by which Γ acts on L is very rigid. This structure transfers to the modules in $C\Gamma$, partially due to the fact that L is *coherent* (a condition closely related to a ring being Noetherian). In the Noetherian case, for any finitely presented module M over such a ring R, there always exists a filtration such that its subquotients are isomorphic to $R/\mathfrak{p}$ for $\mathfrak{p} \subset R$ prime. For the Lazard Ring, not only can we find a similar filtration $F_1M \subset F_2M \subset \ldots \subset F_kM = M$ such that the subquotients are isomorphic to quotients by prime ideals, but the filtration is compatible with the Γ action.

Theorem 2.8 – Landweber Filtration Theorem Every module $M \in C\Gamma$ can be filtered by a finite chain of submodules as above such that:

1. Each submodule F_iM is invariant under the action of Γ
2. Each subquotient $F_{i+1}M/F_iM$ is isomorphic to (a suspension of) $L/I_{p,n}$ for n finite.

This is a nice fact, but it is not immediately obvious why one might care about the existence of such filtrations. As we noted in the beginning of this section, the rich structure of $C\Gamma$ is one of the reasons for the results central to this reading group. The above theorems allow us to prove the following facts

Proposition 2.3 For a p-local $M \in C\Gamma$ and $x \in M$, the following are true:

1. If $v_n^i x = 0$ for some i, then $v_{n-1}^j x = 0$.
2. If x is a nontrivial element of M, then $\exists n : \forall k, v_n^k x \neq 0$
3. If each element in M is annhilated by a power of v_{n-1}, then multiplication by v_n commutes with the Γ action
4. If some element of M is *not* annhilated by any power of v_{n-1}, then multiplication by v_n does *not* commute with the Γ action.

Remark 2.2 The above statements should be interepreted as algebraic analogues of the following toplogical facts concerning the Morava K-Theories $K(n)_*$

1. For X a p-local finite CW complex, $\overline{K(n)_*(X)}$ vanishing imples that $\overline{K(n-1)_*(X)}$ vanishies.
2. If, moreover, X is not contractible then there exists an n such that $\overline{K(n)_*(X)} \cong K(n)_*(*) \otimes \overline{H}_*(X; \mathbb{Z}/(p))$.

and the third statement should be interpreted as an algebraic analogue of the *Periodicity Theorem.*