Formal Group Laws

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Hopf Algebroid

Definition

A Hopf algebroid over a commutative ring K is the cogroupoid object in the category of commutative K-algebras $\mathbf{CRing_K}$.

In other words, it is a pair (A, Γ) of commutative K-algebras s.t. $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(\Gamma, B)$ form the objects and morphisms of a groupoid, for every commutative K-algebra B, with the following structure maps:

- $\eta_L : A \to \Gamma$ source,
- $\eta_R : A \to \Gamma$ target,
- $\Delta : \Gamma \to \Gamma \otimes_A \Gamma$ *composition*,
- $\varepsilon : \Gamma \to A$ identity,
- $c : \Gamma \to \Gamma$ inverse.

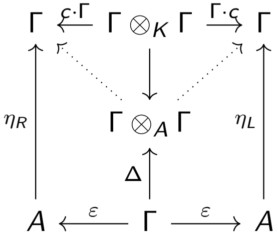
Γ is a A-bimodule made by η_L and η_R. We also require Δ and ε to be A-bimodule maps.

Axioms for Structure Maps

These structure maps must satisfy the following identities:

1. $\varepsilon\eta_L = \varepsilon\eta_R = 1_A$, the identity map on A. (The source and target of an identity morphism are the object on which it is defined.)
2. $(\Gamma \otimes \varepsilon)\Delta = (\varepsilon \otimes \Gamma)\Delta = 1_\Gamma$. (Composition with the identity leaves a morphism unchanged.)
3. $(\Gamma \otimes \Delta)\Delta = (\Delta \otimes \Gamma)\Delta$. (Composition of morphisms is associative.)
4. $c\eta_R = \eta_L$ and $c\eta_L = \eta_R$. (Inverting a morphism interchanges source and target.)
5. $cc = 1_\Gamma$. (The inverse of the inverse is the original morphism.)
6. Maps exist which make the following diagram commute where $c \cdot \Gamma(\gamma_1 \otimes \gamma_2) = c(\gamma_1)\gamma_2$ and $\Gamma \cdot c(\gamma_1 \otimes \gamma_2) = \gamma_1 c(\gamma_2)$.

Axioms for Structure Maps (Continued)



The composition of a morphism with its inverse on either side gives an identity morphism.

Variation and Motivation

Definition

A **graded commutative Hopf-algebroids** is a pair of graded commutative algebras (A, Γ) with graded commutative structure maps.

Definition

A graded Hopf algebroid (A, Γ) is said to be **connected** if the right and left sub A-modules $\Gamma_0 \hookrightarrow \Gamma$ are both isomorphic to A

Example

$(\pi_*(E), E_*(E))$ is a Hopf algebroid for suitable E, e.g. $E = MU, BP, H\mathbb{Z}/2, \ldots$

Universal Formal Group Laws and Strict Isomorphisms

Definition

Let R be a commutative ring with unit. A formal group law F over R is a power series $F(x, y) \in R[[x, y]]$ satisfying

1. $F(0, x) = F(x, 0) = x$,

Universal Formal Group Laws and Strict Isomorphisms

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2. $F(x, y) = F(y, x)$,
3. $F(F(x, y), z) = F(x, F(y, z))$.

Universal Formal Group Laws and Strict Isomorphisms

Here are some easy propositions.

Proposition

$$F(x, y) \equiv x + y \mod (x, y)^2$$

Proposition

If F is a formal group law over R then there is a power series $i(x) \in R[[x]]$ (called the formal inverse) such that $F(x, i(x)) = 0$.

Here are some examples:

Example

1. $F_a(x, y) = x + y$, the additive formal group law.
2. $F(x, y) = x + y + uxy$ (where u is a unit in R), the multiplicative formal group law, so named because $1 + uF = (1 + ux)(1 + uy)$.
3. $F(x, y) = (x + y)/(1 + xy)$.
4. $F(x, y) = \left(x\sqrt{1 - y^4} + y\sqrt{1 - x^4}\right) / \left(1 + x^2y^2\right)$, a formal group law over $\mathbf{Z}[1/2]$.

Universal Formal Group Laws and Strict Isomorphisms

Definition

Let F and G be formal group laws over R.

- A homomorphism from F to G is a power series $f(x) \in R[[x]]$ with constant term 0 such that $f(F(x,y)) = G(f(x), f(y))$,
- isomorphism if invertible, i.e. if $f'(0) \in R^x$,
- strict isomorphism if $f'(0) = 1$.

Universal Formal Group Laws and Strict Isomorphisms

Theorem

Let F be a formal group law and then there exists a logarithm $f(x) \in$ over $R \otimes \mathbf{Q}$.

In fact, it can be constructed explicitly:

$$f(x) = \int_0^x \frac{dt}{F_2(t, 0)}$$

where $F_2(x, y) = \partial F / \partial y$.

Universal Formal Group Laws

Definition (Theorem)

There is a ring L (called the Lazard ring) and a formal group law

$$F(x,y) = \sum a_{i,j} x^i y^j$$

defined over it such that for any formal group law G over any commutative ring with unit R there is a unique ring homomorphism $\theta : L \to R$ such that $G(x,y) = \sum \theta(a_{i,j}) x^i y^j$

It's useful to introduce a grading $|a_{i,j}| = 2(i+j-1)$.

Lemma

1. $L \otimes \mathbf{Q} = \mathbf{Q}[m_1, m_2, \ldots]$ *with* $|m_i| = 2i$ *and* $F(x,y) = f^{-1}(f(x)+f(y))$ *where* $f(x) = x + \sum_{i>0} m_i x^{i+1}$.
2. *Let* $M \subset L \otimes \mathbf{Q}$ *be* $\mathbf{Z}[m_1, m_2, \ldots]$. *Then* $\operatorname{im} L \subset M$.

Universal Formal Group Laws

Definition

R graded connected (e.g., $L \otimes \mathbf{Q}$) the group of indecomposables QR is I/I^2 where $I \subset R$ is the ideal of elements of positive degree.

It's a direct summand in the associated graded ring of R w.r.t I.

Theorem (Lazard)

1. $L = \mathbf{Z}[x_1, x_2, \ldots]$ *with* $|x_i| = 2i$ *for* $i > 0$.
2. x_i *can be chosen so that its image in* $QL \otimes \mathbf{Q}$ *is*

$$\begin{cases} pm_i & \text{if } i = p^k - 1 \text{ for some prime } p \\ m_i & \text{otherwise.} \end{cases}$$

3. L *is a subring of* M.

This important theorem is a consequence of the following lemmas.

Universal Formal Group Laws

Definition

$$B_n(x,y) = (x+y)^n - x^n - y^n$$

$$C_n(x,y) = \begin{cases} B_n/p & \text{if } n = p^k \text{ for some prime } p \\ B_n & \text{otherwise.} \end{cases}$$

C_n is integral and is not divisible by any prime number.

Universal Formal Group Laws

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Lemma (Comparison)

Let F and G be two formal group laws over R such that $F \equiv G \bmod (x,y)^n$. Then $F \equiv G + aC_n \bmod (x,y)^{n+1}$ for some $a \in R$.

Lemma

(a) In $QL \otimes \mathbf{Q}$, $a_{i,j} = -\binom{i+j}{j} m_{i+j-1}$. (b) QL is torsion-free.

Universal Formal Group Laws

Now, let's view formal group laws over arbitrary commutative rings with unit all together.

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Definition

Let R be a commutative ring with unit. Then $FGL(R)$ is the set of formal group laws over R and $SI(R)$ is the set of triples (F, f, G) where $F, G \in FGL(R)$ and $f : F \to G$ is a strict isomorphism, i.e., $f(x) \in R[[x]]$ with $f(0) = 0, f'(0) = 1$, and $f(F(x, y)) = G(f(x), f(y))$. We call such a triple a matched pair.

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In fact, they are covariant functors.

Theorem

FGL(-) and $SI(-)$ are covariant functors on the category of commutative rings with unit. $FGL(-)$ is represented by the Lazard ring L and $SI(-)$ is represented by the ring $LB = L \otimes \mathbf{Z}[b_1, b_2, \ldots]$. In the grading introduced above, $|b_i| = 2i$.

Universal Formal Group Laws

Theorem

In the Hopf algebroid (L, LB), $\varepsilon : LB \to L$ is defined by $\varepsilon(b_i) = 0; \eta_L : L \to LB$ is the standard inclusion while $\eta_R : L \otimes \mathbf{Q} \to LB \otimes \mathbf{Q}$ is given by

$$\sum_{i \geq 0} \eta_R(m_i) = \sum_{i \geq 0} m_i \left(\sum_{j \geq 0} c(b_j) \right)^{i+1}$$

where $m_0 = b_0 = 1; \sum_{i \geq 0} \Delta(b_i) = \sum_{j \geq 0} \left(\sum_{i \geq 0} b_i \right)^{j+1} \otimes b_j$; and $c : LB \to LB$ is determined by $c(m_i) = \eta_R(m_i)$ and $\sum_{i \geq 0} c(b_i) \left(\sum_{j \geq 0} b_j \right)^{i+1} = 1$

These are the structure formulas for $MU_*(MU)$.

p-typical Formal Group Laws

We have obtained important theorems for general formal group laws. But the tools become sharper when we localize at p.
The definition below it's just for convenience.

Definition (Proposition)

Let F be a formal group law over R. If x and y are elements in an R-algebra A which also contains the power series $F(x, y)$, let

$$x +_F y = F(x, y).$$

This notation may be iterated, e.g., $x +_F y +_F z = F(F(x, y), z)$. Similarly, $x -_F y = F(x, i(y))$. For nonnegative integers n, $[n]_F(x) = F(x, [n-1]_F(x))$ with $[0]_F(x) = 0$. (The subscript F will be omitted whenever possible.) $\sum^F(\ )$ will denote the formal sum of the indicated elements.

p-typical Formal Group Laws

Definition (Proposition)

If the formal group law F is defined over a K algebra R where K is a subring of $\mathbf{Q}$, then for each $r \in K$ there is a unique power series $[r]_F(x)$ such that

- if r is a nonnegative integer, $[r]_F(x)$ is the power series defined above,
- $[r_1 + r_2]_F(x) = F([r_1]_F(x), [r_2]_F(x))$,
- $[r_1 r_2]_F(x) = [r_1]_F([r_2]_F(x))$.

p-typical Formal Group Laws

Definition

Suppose q is a natural number that is invertible in R, then we define

$$f_q(x) := [\frac{1}{q}]_F \sum_{i=1}^{q} {}^F \zeta^i x$$

where ζ is the primitive q-th root of unity.

Definition

A formal group law F over a $\mathbf{Z}_{(p)}$-algebra is p-typical if $f_q(x) = 0$ for all primes $q \neq p$.

In fact, it can be simplified when the algebra is torsion-free.

Definition

A formal group law over a torsion-free $\mathbf{Z}_{(p)}$-algebra is p-typical if its logarithm has the form $\sum_{i\geq 0} \ell_i x^{p^i}$ with $\ell_0 = 1$

p-typical Formal Group Laws

Theorem (Cartier)

Every formal group law over a $\mathbf{Z}_{(p)}$-algebra is canonically strictly isomorphic to a p-typical one.

Proof.

Suffices to construct a strict isomorphism $f(x) = \sum f_i x^i \in L \otimes \mathbf{Z}_{(p)}[[x]]$ from the image of F over $L \otimes \mathbf{Z}_{(p)}$ to a p-typical formal group law F'. Now if G is a formal group law over a $\mathbf{Z}_{(p)}$-algebra R induced by a homomorphism $\theta : L \otimes \mathbf{Z}_{(p)} \to R, g(x) = \sum \theta(f_i) x^i \in R[[x]]$ is a strict isomorphism from G to a p-typical formal group law G'. Thus, let mog(x) be the logarithm for F', then $mog(x) = log(f^{-1}(x))$ and consider $f^{-1}(x) = \sum^F_{p \nmid q} [\mu(q)]_F (f_q(x))$, where $\mu(x)$ is the Mobius function. □

p-typical Formal Group Laws

Definition (Theorem)

Let $V = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ with $|v_n| = 2(p^n - 1)$. Then there is a universal p-typical formal group law F defined over V; i.e., for any p typical formal group law G over a commutative $\mathbf{Z}_{(p)}$-algebra R, there is a unique ring homomorphism $\theta : V \to R$ such that $G(x, y) = \theta(F(x, y))$. Moreover the homomorphism from $L \otimes \mathbf{Z}_{(p)}$ to V corresponding to this formal group law is surjective, i.e., V is isomorphic to a direct summand of $L \otimes \mathbf{Z}_{(p)}$.

To construct V, note the canonical isomorphism f constructed before corresponds to an endomorphism ϕ of $L \otimes \mathbb{Z}_p$, given by

$$\phi(m_i) = \begin{cases} m_i & \text{if } i = p^k - 1 \\ 0 & \text{otherwise.} \end{cases}$$

And it is **idempotent**, i.e. $\phi^2 = \phi$. Let $V := image(\phi)$.

p-typical Formal Group Laws

Theorem

In the Hopf algebroid (V,VT),

1. $V = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ *with* $|v_n| = 2(p^n - 1)$,
2. $VT = V \otimes \mathbf{Z}_{(p)}[t_1, t_2, \ldots]$ *with* $|t_n| = 2(p^n - 1)$, *and*
3. $\eta_L : V \to VT$ *is the standard inclusion and* $\varepsilon : VT \to V$ *is defined by* $\varepsilon(t_i) = 0, \varepsilon(v_i) = v_i$ *Let* $\ell_i \in V \otimes \mathbf{Q}$ *denote the image of* $m_{p^i-1} \in L \otimes \mathbf{Q}$ *(see A2.1.9). Then*
4. $\eta_R : V \to VT$ *is determined by* $\eta_R(\ell_n) = \sum_{0 \le i \le n} \ell_i t_{n-i}^{p^i}$ *where* $\ell_0 = t_0 = 1$,
5. Δ *is determined by* $\sum_{i,j \ge 0} \ell_i \Delta(t_j)^{p^i} = \sum_{i,k,j \ge 0} \ell_i t_j^{p^i} \otimes t_k^{p^{i+j}}$, *and*
6. c *is determined by* $\sum_{i,j,k \ge 0} \ell_i t_j^{p^i} c(t_k)^{p^{i+j}} = \sum_{i \ge 0} \ell_i$.

The forgetful functor induces a surjection of Hopf algebroid $(L \otimes \mathbb{Z}_{(p)}, LB \otimes \mathbb{Z}_{(p)}) \to (V, VT)$.

Classification

Lemma

Let F be a formal group law over a commutative $\mathbf{F}_p$ -algebra R and let $f(x)$ be a nontrivial endomorphism of F. Then for some n, $f(x) = g\left(x^{p^n}\right)$ with $g'(0) \neq 0$. In particular f has leading term ax^{p^n}.
When R is a perfect field K, we can replace $g\left(x^{p^n}\right)$ by $h(x)^{p^n}$ with $h'(0) \neq 0$.

Definition

A formal group law F over a commutative $\mathbf{F}_p$-algebra R has height n if $[p]_F(x)$ has leading term ax^{p^n}. If $[p]_F(x) = 0$ then F has height ∞.

Classification

- A natural question: can every height be realized by a formal group law over such R? Moreover, since height is invariant under isomorphism, is the converse true? If not, where does it fail?

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- Answer: Yes. Yes in certain field. No in general. In fact, every formal group law attached to a nonsingular elliptic curve always has height 1 or 2. [7.5, Silverman]

Classification

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- Answer: Yes. Yes in certain field. No in general. In fact, every formal group law attached to a nonsingular elliptic curve always has height 1 or 2. [7.5, Silverman]
- The following definition and theorem give a more precise answer.

Classification

Definition

$F_\infty(x, y) = x + y$. For a natural number h let F_n be the p-typical formal group law (of height n) induced by the homomorphism $\theta : V \to R$ defined by $\theta(v_n) = 1$ and $\theta(v_i) = 0$ for $i \neq n$.

Theorem

Let K be a separably closed field of characteristic $p > 0$. A formal group law G over K of height n is isomorphic to F_n.

Classification

In order to do some concrete examples, I need to talk about the generators. Let $l_i \in V \otimes \mathbf{Q}$ be the image of $m_i \in L \otimes \mathbf{Q}$.

Definition (Araki's Generators)

$p\ell_n = \sum_{0 \le i \le n} \ell_i v_{n-i}^{p^i}$

For example,

$$\ell_1 = \frac{v_1}{p - p^p}, \quad \left(p - p^{p^2}\right)\ell_2 = v_2 + \frac{v_1^{1+p}}{p - p^p},$$

$$\left(p - p^{p^3}\right)\ell_3 = v_3 + \frac{v_1 v_2^p}{p - p^p} + \frac{v_2 v_1^{p^2}}{p - p^{p^2}} + \frac{v_1^{1+p+p^2}}{(p - p^p)\left(p - p^{p^2}\right)},$$

Proposition

$$[p]_F(x) = \sum_{i \ge 0}^{F} v_i x^{p^i}$$

Classification

Let $l_i \in V \otimes \mathbf{Q}$ be the image of $m_{p^i-1} \in L \otimes \mathbf{Q}$. For example,

$$\ell_1 = \frac{v_1}{p - p^p}, \quad \left(p - p^{p^2}\right)\ell_2 = v_2 + \frac{v_1^{1+p}}{p - p^p}$$

Example

- $F(x, y) = x + y, [p]_F(x) = 0$ for all p so F has height ∞.
- $F(x, y) = x + y + uxy, [p]_F(x) = u_{p-1}x^p$ so F has height 1 .
- $F(x, y) = \frac{x+y}{1+xy}$, F is isomorphic over $\mathbf{Z}_{(2)}$ to the additive formal group law, so its height at $p = 2$ is ∞. Its logarithm is

$$tanh^{-1}(x) = \sum_{i \geq 0} \frac{x^{2i+1}}{2i+1}$$

so for each odd prime p we have $\ell_1 = m_{p-1} = 1/p$, so $v_1 \neq 0 \bmod p$. Thus it has height 1.

Classification

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- Let E_n be the algebra obtained from $W(\mathbf{F}_q)$ $(q = p^n)$, so-called Witt ring, by adjoining an indeterminate S and setting $S^n = p$ and $Sw = w^\sigma S$ for $w \in W(\mathbf{F}_q)$ where σ is the lifting of the Frobenius automorphism on $\mathbf{F}_q$.

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Now, the following theorems explicitly characterize the endomorphism ring of F_n.

Endomorphism Rings

Theorem (Dieudonné, Lubin)

Let K be a field of characteristic p containing $\mathbf{F}_q$, with $q = p^n$. Then the endomorphism ring of the formal group law F_n over K is isomorphic to E_n. The generators ω and S correspond to endomorphisms $\bar{\omega}x$ and x^p, respectively.

Theorem

Let R be a commutative $\mathbf{F}_p$-algebra. Then the endomorphism ring of the additive formal group law F_∞ over R is the noncommutative power series ring $R\langle\langle S\rangle\rangle$ in which $Sa = a^p S$ for $a \in R$. The elements a and S correspond to the endomorphisms ax and x^p, respectively.

Moduli Stacks of Formal Groups

- Let $L := \mathbb{Z}[a_{i,j}]/I$ be the Lazard ring defined before. Let $G := Spec\mathbb{Z}[b_1, b_2, \ldots]$.

Moduli Stacks of Formal Groups

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- We can equivalently view an affine scheme as a functor from **CRing** $\rightarrow$ **Set** (G(R):=Hom(Spec(R),G)), in fact "one of the principal goals in Grothendieck's work on schemes was to find a characterization of scheme-functors by weak general properties that could often be checked in practice and so lead to many existence theorems in algebraic geometry. (like Brown's theorem in (Hot))." [Chapter I,6,Mumford-Oda]

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- Thus, assume G is an affine group scheme, to give an action of G on spec L is equivalent to give an action of $G(R) = \{g \in R[[t]] | g(t) = t + b^1 t^2 + \ldots\}$ on $SpecL(R) \cong FGL(R)$, given by $g \cdot f(x, y) = g^{-1} f(g(x), g(y))$. This defines a quotient stack, $SpecL//G$.

- We can expand G a little bit. Note that $r \in R^x$ acts on $FGL(R)$ by sending f to $r^{-1}f(rx, ry)$. So we can define $G^+(R) := \{g \in R[[x]] : g(t) = b_0t + b_1t^2 + \ldots, b_0 \in R^x\}$. It can be identified by the semidirect product of the two groups and acts on $SpecL$ by substitution.

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- This forms a quotient stack, $SpecL//G^{+}$, which is the moduli stack of formal groups.

Moduli Stacks of Formal Groups

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Moduli Stacks of Formal Groups

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- This defines a functor $\mathbf{CRing}_R \to \mathbf{Ab}$.
- It turns out, not surprisingly, this is an equivalent way to think of formal group laws, i.e. the natural transformation between two such functors is precisely represented by a power series. And it acts the same as our definition of homomorphism between FGLs.

Moduli Stacks of Formal Groups

- But there is one problem. The collection of such functors does not satisfy descent in R, which is similar to the gluing conditions of sheaves over Zariski topology.
- So it is useful to expand the definition a little bit.
- First, the functor $\mathcal{G}$ is a sheaf with respect to the Zariski topology. In other words, if A is a commutative R-algebra with a pair of elements x and y such that $x + y = 1$, then $\mathcal{G}(A)$ can be described as the subgroup of $\mathcal{G}\left(A\left[\frac{1}{x}\right]\right) \times \mathcal{G}\left(A\left[\frac{1}{y}\right]\right)$ consisting of pairs which have the same image in $\mathcal{G}\left(A\left[\frac{1}{xy}\right]\right)$.

- Second, The functor $\mathcal{G}$ is a coordinatizable formal group law locally with respect to the Zariski topology. That is, we can choose elements $r_1, r_2, \ldots, r_n \in R$ such that $r_1 + \cdots + r_n = 1$, such that each of the composite functors

$$\mathrm{Alg}_{R\left[\frac{1}{r_i}\right]} \to \mathrm{Alg}_R \to \mathrm{Ab}$$

has the form $\mathcal{G}_f$ for some formal group law $f \in R\left[\frac{1}{r_i}\right][[x, y]]$.

Moduli Stacks of Formal Groups

Now we can define the lie algebra of the functor (tangent space of a functor), and explore all sorts of relations between them. For a clear and detailed discussion of these, see [Lurie].

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