WEEK 1 NOTES, REVIEW OF SOME HOMOTOPY THEORY

Contents

Safe to assume some understanding of: homotopy equivalence, basic category theory, CW-complexes, homotopy groups, generalized (co)homology theories. (Loop spaces? Suspensions? Brown representability?)

1. Space-Level Stuff

1.1. Suspensions, Loops.

Given a based space (X, x_0) we define its *reduced suspension* and *loop space*

$$\Sigma X := X \times I / \sim \quad \text{and} \quad \Omega X := \mathrm{Map}_*(S^1, X),$$

Where $\sim$ identifies all the points $X \times \{0\}$, $X \times \{1\}$, and $\{x_0\} \times I$, and where $\mathrm{Map}_*(C, D)$ denotes the space of continuous, based maps $C \to D$ with the compact-open topology. Given a based map $f : X \to Y$, we define $\Sigma f, \Omega f$ between the suspension/loop spaces in the obvious way. These describe functors

$$\Sigma, \Omega : \mathrm{Top}_* \to \mathrm{Top}_* \quad \text{and} \quad \Sigma, \Omega : Ho(\mathrm{Top})_* \to Ho(\mathrm{Top})_*$$

Definition 1.1. Suppose as given a map $f : \Sigma X \to Y$. Each point $x \in X$ determines a loop $f(\{x\} \times I)$ in Y. This describes a based map $\hat{f} : X \to \Omega Y$, the *adjoint* to f.

Proposition 1.2. *The mapping $f \mapsto \hat{f}$ describes isomorphisms*

$$\mathrm{Map}_*(\Sigma^i X, Y) \to \mathrm{Map}_*(X, \Omega^i Y) \quad \textit{and} \quad [\Sigma^i X, Y] \to [X, \Omega^i Y].$$

(Corollary: $\pi_{n+k} Y \cong \pi_n \Sigma^k Y$.) In fact, this latter isomorphism is natural in X and Y, so we get $\Sigma \dashv \Omega$.

1.2. Cofibers.

Given a map $f : X \to Y$, we define its *cofiber* or *mapping cone* to be the space

$$C_f := ((X \times I) \cup Y) / \sim,$$

Where $\sim$ collapses $X \times \{0\}$ and glues $X \times \{1\}$ to $f(X) \subseteq Y$ in the obvious way. If, in addition, everything is based, then we additionally collapse $\{x_0\} \times I$ to a basepoint. The cofiber comes with a canonical inclusion

$$i_f : Y \hookrightarrow C_f.$$

It is easy to see that $C_{i_f} \simeq \Sigma X$, thus we may form a map $j_f : C_f \to \Sigma X$.

Definition 1.3. Given a map $f : X \to Y$, the *cofiber sequence generated by f* is the sequence of maps

$$X \xrightarrow{f} Y \xrightarrow{i_f} C_f \xrightarrow{j_f} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{i_{\Sigma f}} C_{\Sigma f} \xrightarrow{j_{\Sigma f}} \Sigma^2 X \longrightarrow \cdots$$

A *cofiber sequence* is a sequence of maps arising this way.

We think of a cofiber sequence as a long exact sequence of spaces, for up to homotopy equivalence, any two consecutive maps in a cofiber sequence are the composite of an inclusion and a quotienting of its image. Cofiber sequences induce a long exact sequence of homotopy classes of maps.

Proposition 1.4. *Suppose $f : X \to Y$ is a based map of path-connected CW complexes. For any based space Z, the cofiber sequence generated by f induces a sequence*

$$\cdots \longrightarrow [\Sigma C_f, Z] \longrightarrow [\Sigma Y, Z] \longrightarrow [\Sigma X, Z] \longrightarrow [C_f, Z] \longrightarrow [Y, Z] \longrightarrow [X, Z]$$

This is a long exact sequence of pointed sets, or of groups to the left $[\Sigma X, Z]$.

Remark 1.5. The map $\hat{f} : X \to C_f$ given by $x \mapsto (x, 1)$ is a cofibration. Also, there is an obvious deformation retract $C_f \to Y$ (in particular, a homotopy equivalence). The composite $X \to C_f \to Y$ equals f. So, up to a homotopy equivalence, we can replace maps f with cofibrations $\hat{f}$.

Remark 1.6. There is a dual theory for *fibrations*. I won't develop this. It is worth reading Concise about this stuff.

2. Spectra-Level Stuff

2.1. **Basic definitions.** A *spectrum* X is a collection of pointed spaces $\{X_i\}$ and maps $\{\Sigma X_i \to X_{i+1}\}$. If all the adjoint maps $X_i \to \Omega X_{i+1}$ are weak equivalences, we call X an *Ω-spectrum*. One good way to motivate spectra (specifically Ω-spectra) is that they are the objects needed to represent reduced cohomology theories. They are also a good framework to think about stable phenomena.

Definition 2.1. Now here is some terminology. Let X be a spectrum.

- *Homotopy groups*: $\pi_k X := \operatorname{colim} \pi_{n+k} X_n$.
- *Generalized homology*: Given a reduced homology theory $\bar{E}_*$ we define $E_k(X) := \operatorname{colim} E_{n+k} X_n$. Another way to go about this is to say, for a *spectrum* E, to define $E_n X := \pi_n(XE)$.
- *Generalized cohomology*: Given a spectrum E, we define $E^n(X) := [\Sigma^n X, E]$.
- A spectrum is *connective* if its homotopy groups are bounded below.
- A spectrum has *finite type* if its homotopy groups are finitely-generated.
- A spectrum X is *finite* if there is a finite CW-complex C such that $X_n = \Sigma^n C$ and the structure maps are the identities.

Definition 2.2. A *sequential map of spectra* $f : E \to F$ is a collection of maps $f_n : E_n \to F_n$ such that Σf_n and f_{n+1} commute with the structure maps.

The notion of a sequential map is too restrictive. There are things we want as maps or spectra that are not sequential maps, see e.g. p. 130 of the orange book. Ravenel defines a map of spectra $X \to Y$ as a map from X to a homotopy-equivalent replacement of Y by an Ω-spectrum.

Definition 2.3. A *map of spectra* $f : E \to F$ is a collection of maps

$$f_n : E_n \to \operatorname{colim} \Omega^k F_{n+k}$$

Satisfying $f_n = \Omega f_{n+1}$.

(Say something about eventually-defined maps?)

One point of spectra is to let us do algebra that we wanted to do with spaces. The smash product is an important structure for doing this. Getting the definition right is very hard, let's keep with Ravenel and give the following limited definition.

Definition 2.4. The *naive smash product* of two spectra E, F is the spectrum given by

$$(E \wedge F)_{2n} := E_n \wedge F_n,$$
$$(E \wedge F)_{2n+1} := \Sigma E_n \wedge F_n.$$

Definition 2.5. Somewhat more easy, the *smash product of a spectrum and a space* of a spectrum X and a pointed space A is given by $(X \wedge A)_n := X_n \wedge A$.

Definition 2.6. The *homotopy category of CW-spectra* has

- As objects, spectra whose spaces have CW homotopy type; and
- As morphisms, homotopy classes of maps of spectra. (Haven't defined homotopy. It's more-or-less what it should be: a map $X \wedge I^+ \to Y$, the domain being the "cylinder spectrum," such that the original maps are compatible through the inclusions to endpoints.)

Thus we have categories of CW spectra and its homotopy category, denoted Sp_{CW} and SHC.

2.2. **Examples of spectra.**

- Given a spectrum X, its i-th suspension is given as $(\Sigma^i X)_n := X_{n+i}$.
- Given a space X, its *suspension spectrum* $\Sigma^\infty X$ is defined by $(\Sigma^\infty X)_n = \Sigma^n X$, with structure maps the identity. This defines a functor $Top_* \to \mathsf{SHC}$.
- The *sphere spectrum* $\mathbb{S}$ is the suspension spectrum of S^0, i.e. $\Sigma^\infty S^0$. One has $\mathbb{S}_n \cong S^n$ and $\pi_k \mathbb{S} \cong \pi_k^s$.
- For an abelian group G, we can form the *Eilenberg-Maclane spectrum* by $(HG)_n := K(G, n)$. The structure maps come from the homotopy equivalences $K(G, n) \simeq \Omega K(G, n+1)$ and the loop-suspension adjunction.

- For an abelian group G, the *Moore spectrum of G*, written MG or SG, is characterized as (1) having only one nonzero homotopy group $\pi_0 = G$, and (2) zero singular homology in positive degrees. For a construction see Adams p. 217. I think we write $\mathbb{S}/(p)$ for the Moore spectrum of $\mathbb{Z}/p$?

2.3. **Homotopy direct limits.** Just as for pointed spaces, the *coproduct* of any collection of spectra $\{Z_i\}$ is built out of wedges: one has $\coprod_i Z_i \cong \bigvee_i Z_i$ where the latter has $\bigvee_i (Z_i)_n$ as its n-th space. The formation of coproducts commutes with π_*, smashing with a spectrum $- \wedge E$, and E_*.

Denote by $\mathbf{X}$ the following directed system of spectra.

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$$

We want to form the direct limit of this system. If the X_i are replaced by abelian groups A_i, this is easy: we take the *shift map* $s : \bigoplus_i A_i \to \bigoplus_i A_i$ given by $a_i \mapsto a_i - f(a_i)$ and take its cokernel. It's plausible that there is an analogous *shift map* $s : \bigvee X_i \to \bigvee X_i$ that induces the shift map on homology. (To-do: what is the map? Ravenel just says it exists. Maybe check blue book?) This induces the shift map on homology groups. Then we want its "cokernel." This amounts to taking cofibers.

Definition 2.7. Let $f : X \to Y$ be a map of spectra. Denote by C_f or $Y \cup_f CX$ the *cofiber* of f. Imprecisely, it's what you think it is: it is constructed by replacing f by a collection of cellular maps and forming cofibers at every n-th level. (We need to be imprecise because we're avoiding terminology; see Adams p. 172.)

Definition 2.8. We define the *homotopy direct limit* $\operatorname{hocolim} \mathbf{X}$ of $\mathbf{X}$ as the cofiber of the shift map:

$$\operatorname{hocolim} \mathbf{X} := \operatorname{cofib}\left(s : \bigvee_i X_i \to \bigvee_i X_i \right).$$

Proposition 2.9. *Homology E_* commutes with homotopy direct limits.*

Proposition 2.10. *The homotopy direct limit is not a categorical colimit. Rather, there is a long exact sequence relating the graded pieces*[1] *of $\prod_i [X_i, Y]$ and $[\operatorname{hocolim} \mathbf{X}, Y]$. (Written out on Ravenel, p. 142.) The point: if a set of maps $\{g_i : X_i \to Y\}$ is compatible, we get a map $g : \operatorname{hocolim} \mathbf{X} \to Y$ through which they factor, but $[g]$ may not be uniquely determined.*

Proposition 2.11 (A.5.8 in Ravenel)**.** *Let X be any CW spectrum. Recall that a spectrum is called finite if it is equivalent to a finite CW complex. A finite subspectrum of X is a map $F \to X$ where F is a finite spectrum. These form a category in the obvious way. In fact, this category is directed, so we may form its homotopy colimit $\operatorname{hocolim} F_\alpha$. By construction, there is a canonical map*

$$\lambda : \operatorname{hocolim} F_\alpha \to X.$$

This map is a weak homotopy equivalence. So, any CW spectrum is canonically the hocolim of its finite subspectra.

It's worth looking at the (not hard) proof of this proposition.

Proposition 2.12 (Milnor SES)**.** *For any spectrum E and each integer n we have a short exact sequence*

$$0 \to \lim\nolimits^1_{\leftarrow} E^{n-1}(X_i) \to E^n(\operatorname{hocolim} \mathbf{X}) \to \lim\nolimits_{\leftarrow} E_n(X_i) \to 0.$$

Corollary 2.13. *If X, E are spectra such that the inverse system $\{E^{n-1}X_n\}_n$ is Mittag-Leffler, then two maps $X \to E$ are homotopic if their components $X_n \to E_n$ are(?)*

2.4. **Homotopy inverse limits.** The *product* of a collection of spectra $\{X_\alpha\}$ is not in general as easy to construct as the coproduct. Ravenel constructs (A.4.3) it using Brown representability: *the functor $\prod_\alpha [-, X_\alpha]$ satisfies the E-S axioms and the wedge axiom, hence is represented by a spectrum christened $\prod_\alpha X_\alpha$, and this is a categorical product.* If the collection $\{X_\alpha\}$ is finite, then this product coincides with the coproduct. If the X_α are Ω-spectra, then the product can be explicitly constructed out of cartesian products of underlying spaces (see the proof of A.4.3), as one might expect.

[1] This is something I think also gets a little lost when we do away with a full-frontal definition of maps of spectra. See Adams, p. 159 and onward.

Proposition 2.14. *Unlike for coproducts, if $\{X_\alpha\}$ is an infinite collection of spectra, taking the product does not commute with smashing in general, in particular E-homology for a spectrum E. Here are two exceptions.*

(1) If E and the X_α are connective, and for each n one has $\pi_n X_\alpha = 0$ for all but finitely many α, then

$$E_*(\prod X_\alpha) = \bigoplus (E_* X_\alpha).$$

(2) Likewise if E is finite.

Denote by $\mathbf{X}$ the following inverse system of spectra.

$$\cdots \xrightarrow{f_3} X_3 \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1$$

Analogous to the colimit, we will define the homotopy inverse limit as the fiber of a map, defined so that it induces on homology the map whose kernel is the typical inverse limit of a system of abelian groups. (See Ravenel p. 146 for slightly more.)

Proposition 2.15. *There is a shift map*

$$s : \prod_\alpha X_\alpha \to \prod_\alpha X_\alpha$$

Which on homology induces the shift map $(a_1, a_2, \ldots) \mapsto (a_1 - f_1(a_2), a_2 - f_2(a_3), \ldots)$. (???)

Definition 2.16. The *homotopy inverse limit* of $\mathbf{X}$, written $\operatorname{holim}\mathbf{X}$, is defined as

$$\operatorname{holim}\mathbf{X} := \Sigma^{-1}\operatorname{cofib}(s).$$

(The desuspension of the cofiber is the fiber.)

Proposition 2.17. *The homotopy inverse limit is not a categorical limit. But, for a finite spectrum E (e.g., the sphere spectrum) and each integer n, there is a functorial short exact sequence*

$$0 \to \lim\nolimits^1_{\leftarrow} E_{n+1}(X_i) \to E_n(\operatorname{holim}\mathbf{X}) \to \lim\nolimits_{\leftarrow} E_n(X_i) \to 0.$$

2.5. **Structured spectra.** A *ring spectrum* is a spectrum E together with a multiplication $m : E \wedge E \to E$ and a unit $\eta : \mathbb{S} \to E$ such that the three diagrams for unitality/associativity are homotopy commutative. If, in addition, the commutativity diagram is homotopy commutative, we say E is commutative.

Now suppose as given a ring spectrum E. A *E-module spectrum* is a spectrum M with a (homotopy) action by E; that is, a map $\mu : E \wedge M \to M$ such that the following diagrams are homotopy commutative. (I think Ravenel has a typo in his diagram.)

$$\begin{tikzcd} E \wedge E \wedge M \arrow[r, "m\wedge \mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\mu"'] & E \wedge M \arrow[d, "\mu"] \\ E \wedge M \arrow[r, "\mu"'] & M \end{tikzcd} \qquad \begin{tikzcd} M = \mathbb{S} \wedge M \arrow[r, "\eta\wedge \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & M \wedge M \arrow[d, "\mu"] \\ & M \end{tikzcd}$$

The sphere spectrum is the unit for the smash product. As a result, elements in a spectrum's homotopy act upon it.

Definition 2.18. Let E be a ring spectrum. Suppose as given an element $v \in \pi_d E$ represented by a map $f : S^d \to E$. We may consider the composite

$$\Sigma^d E = S^d \wedge E \xrightarrow{f\wedge \mathrm{id}} E \wedge E \to E.$$

We denote this by composite by f also. We have an induced map $\pi_{*-d} E = [S^0, \Sigma^d E] \xrightarrow{f_*} [S^0, E] = \pi_* E$, which turns out to be multiplication by v. Now we define a spectrum $v^{-1}E$ as the homotopy direct limit of multiplication by v^{-1}:

$$v^{-1}E := \operatorname{hocolim}\left(E \xrightarrow{f} \Sigma^{-d} E \xrightarrow{f} \Sigma^{-2d} E \to \cdots\right).$$

Proposition 2.19. *The spectrum $v^{-1}E$ is an E-module spectrum (is there an obvious/canonical map?) and has homology $E_* \otimes_{\mathbb{Z}[v]} \mathbb{Z}[v, v^{-1}]$.*

Also, here is a definition.

Definition 2.20. A ring spectrum E is called *flat* if $E \wedge E$ is equivalent to a wedge of suspensions of E.