THE NILPOTENCE THEOREM

These expository notes sketch a proof of the following result from [1]:

Theorem 0.1 (Devinatz–Hopkins–Smith)**.** *Let R be a connective ring spectrum of finite type and $h_{\mathrm{MU}*} : R_* \to \mathrm{MU}_*R$ the Hurewicz map. Given $\alpha \in \pi_*R$, if $h_{\mathrm{MU}*}(\alpha) \in \mathrm{MU}_*R$ is 0 then α is nilpotent.*

1. Introduction

1.1. **Periodic phenomena.** For fixed primes $p > 2$ Adams constructed self-maps v_1 of the mod p Moore spectrum:

$$\Sigma^{2p-2}S^0/p \xrightarrow{v_1} S^0/p.$$

The map v_1 induces an isomorphism in K-theory, so any iterate v_1^n is non-zero. In fact, one may construct an infinite family in the homotopy group of spheres out of this:

$$S^{n(2p-2)} \to \Sigma^{n(2p-2)}S^0/p \xrightarrow{v_1^n} S^0/p \to S^1.$$

Varying n one gets the order p part of the image of the J-homomorphism. The picture below illustrates the case $p = 3$.

This can be extended for example to the Smith–Toda complex $V(1) := \mathrm{cofib}(v_1)$ for $p \geq 5$, which admits v_2-self maps $\Sigma^{2(p^2-1)}V(1) \xrightarrow{v_2} V(1)$. The effect of this map is no longer an isomorphism on K-theory: one must detect it using something other spectrum.

The nilpotence theorem says that MU is a spectrum that detects all periodicity phenomena: the kernel of the Hurewicz image filters out only nilpotent elements.

1.2. **Versions of nilpotence.** A predecessor to the nilpotence theorem is the following result of Nishida, a special case of the general nilpotence theorem:

Theorem 1.1 (Nishida)**.** *Every element of $\pi_*(S^0)$ for $* > 0$ is nilpotent.*

One may regard the multiplication in $\pi_*(S^0)$ either as coming from the ring structure, the composition of self-maps or smash products. In turn there are three possible generalizations:

(1) For R a ring spectrum, a map $S^m \to R$ is **nilpotent** if $\alpha^n = 0 \in \pi_*R$ for $n \gg 0$.
(2) A self-map $f : \Sigma^d X \to X$ is **nilpotent** if f^n is null for $n \gg 0$.
(3) A map $f : F \to X$ where F is finite is **nilpotent** if $f^{\otimes n} : F^{\otimes n} \to X^{\otimes n}$ is null for $n \gg 0$.

It turns out that MU detects all three forms of nilpotence and they all follow from Theorem 0.1.

1.3. **Relation to other results.** We have $\langle \mathrm{MU}\rangle = \bigoplus_p \langle \mathrm{BP}\rangle$. Furthermore, by a result of Ravenel,

$$\langle \mathrm{BP}\rangle = \langle \mathrm{K}(1)\rangle \oplus \cdots \oplus \langle \mathrm{K}(n)\rangle \oplus \langle \mathrm{BP}\langle n+1\rangle\rangle.$$

As a corollary, one gets the following more refined version of the nilpotence theorem using Morava K-theories:

Theorem 1.2 (Hopkins–Smith)**.** *(1) Let R be a p-local ring spectrum. An element $\alpha \in \pi_* R$ is nilpotent if and only if $\mathrm{K}(n)_*\alpha$ is nilpotent for all $0 \le n \le \infty$.*

(2) A self-map $f : \Sigma^k F \to F$ of the p-localization of a finite spectrum is nilpotent if and only if $\mathrm{K}(n)_* f$ is nilpotent for all $0 \le n < \infty$.

(3) A map $f : F \to X$ from a finite spectrum to a p-local spectrum is smash nilpotent if and only if $\mathrm{K}(n)_* f = 0$ for all $0 \le n \le \infty$.

From there the thick subcategory theorem and the periodicity theorem follow. In fact the thick subcategory theorem turns out to be equivalent to the nilpotence theorem.

2. Overview of the proof

First we record a lemma which is useful throughout.

Lemma 2.1. *For E a ring spectrum, the Hurewicz image of $\alpha \in \pi_m(R)$ in $E_m(R)$ is nilpotent if and only if $E \otimes \alpha^{-1}R \simeq *$. Here $\alpha^{-1}R$ is the colimit*

$$R \xrightarrow{\alpha} \Sigma^{-m} R \xrightarrow{\alpha} \Sigma^{-2m} R \xrightarrow{\alpha} \cdots$$

Remark 2.2. This boils down to the fact that multiplication by α for the R-module structure on $E \otimes R$ is the same as multiplication by the Hurewicz image of α for the ring structure on $E \otimes R$. Notice that it breaks down if E is not a ring spectrum.

The lemma implies that we would be done if $\langle \mathrm{MU}\rangle = \langle S^0\rangle$. This is not true! See Remark 2.8 below.

We give a sketch of the proof, following Devinatz–Hopkins–Smith:

(1) Filter $h_{\mathrm{MU}} : S^0 \to \mathrm{MU}$ by intermediate spaces denoted $X(n)$ as

$$S^0 = X(1) \to X(2) \to \cdots \to X(\infty) = \mathrm{MU}.$$

Here $X(n)$ denotes the Thom spectrum

$$X(n) := \mathrm{Th}(\Omega\mathrm{SU}(n) \to \Omega\mathrm{SU} \simeq \mathrm{BU})$$

where the second map is obtained from Bott periodicity (the only difference in SU and $\mathrm{U} \simeq \mathrm{SU} \times S^1$ is in π_1).

Algebraically:

$$\mathrm{H}_*(\mathrm{MU}) \cong \mathbf{Z}[b_1, b_2, \dots], \quad \mathrm{H}_*(X(n)) \cong \mathbf{Z}[b_1, b_2, \dots, b_{n-1}]$$

where $|b_i| = 2n$.

Remark 2.3. The $X(n)$'s are in fact $\mathbb{E}_2$-ring spectra.

(2) Work p-locally for each prime p. Filter the maps $X(n)_p \to X(n+1)_p$ further by certain spectra G_j:

$$X(n)_p = G_0 \to G_1 \to \cdots \to G_\infty = X(n+1)_p.$$

To define G_j we first set B_k as the homotopy pullback

$$\begin{array}{ccc} B_k & \longrightarrow & \Omega \mathrm{SU}(n+1) \\ \downarrow & & \downarrow \\ J_k S^{2n} & \hookrightarrow & \Omega S^{2n+1} \end{array}.$$

Here $J_k S^{2n}$ is the kth stage of the James construction on S^{2n} (Appendix A), and the right vertical map comes from the fibration

$$\mathrm{SU}(n) \to \mathrm{SU}(n+1) \xrightarrow{e} S^{2n+1}.$$

Now let F_k be the Thom spectrum of $B_k \to \Omega\mathrm{SU}(n+1) \to \mathrm{BU}$ and

$$G_j := (F_{p^j-1})_p.$$

Notice that $B_0 \simeq \mathrm{fib}(\Omega\mathrm{SU}(n+1) \to \Omega S^{2n+1}) \simeq \Omega\mathrm{SU}(n)$, and $G_0 \simeq X(n)_p$. We will omit the lower script p from now on.

Algebraically,

$$\mathrm{H}_*(F_k) \cong \mathbf{Z}[b_1, b_2, \ldots, b_{n-1}]\{1, b_n, b_n^2, \ldots, b_n^k\}$$

as a module over $\mathrm{H}_*(X(n)) \cong \mathbf{Z}[b_1, b_2, \ldots, b_{n-1}]$.

(3) To execute the proof, first pass from $X(\infty)$ to some $X(n)$. Since $\mathrm{colim}_n X(n) \simeq \mathrm{MU}$, we obtain from $h_{\mathrm{MU}}(\alpha) = 0$ that $h_{X(n)}(\alpha) = 0$ for $n \gg 0$. This reduces the nilpotence theorem to the following (backward) inductive claim:

Theorem 2.4. *If $h_{X(n+1)}(\alpha) = 0$ then $h_{X(n)}(\alpha)$ is nilpotent.*

Now we pass from G_∞ to some finite G_j.

Theorem 2.5 ("Step II")**.** *If $h_{X(n+1)}(\alpha)$ is nilpotent, then $G_j \otimes \alpha^{-1}R \simeq *$ for some $j \gg 0$.*

Remark 2.6. There is no ring structure on G_j, so one cannot apply Lemma 2.1. It is true that the Hurewicz image of α is 0 in G_j for $j \gg 0$, but this does not imply that $G_j \otimes \alpha^{-1}R \simeq *$.

Finally, we need the following key result of the entire proof

Theorem 2.7 ("Step III")**.** $\langle G_{j+1}\rangle = \langle G_j\rangle$.

Together with Step I, it implies that $G_0 \otimes \alpha^{-1}R = X(n)_p \otimes \alpha^{-1}R \simeq *$, i.e., $h_{X(n)}(\alpha) = 0$. By induction, $h_{X(0)}(\alpha) = 0$, i.e., $\pi_*(\alpha) = 0$ after p-localizing for any p. This completes the proof.

Remark 2.8. It turns out that $\langle X(p^k - 1)_p\rangle > \langle X(p^k)_p\rangle$, even though in the intermediate steps we do have $\langle X(n)_p\rangle = \langle G_j\rangle$ for all j!

3. Proof sketch for Step II

Below we sketch a proof of the following:

Theorem 3.1. *If $h_{X(n+1)}(\alpha)$ is nilpotent, then $G_j \otimes \alpha^{-1}R \simeq *$ for some $j \gg 0$.*

Without loss of generality replace α by a power α^n so we may assume $h_{X(n+1)}(\alpha) = 0$. This means the representing element $\hat{\alpha} \in E_2^{s,s+d}$ in the $X(n+1)$-based Adams spectral sequence has non-zero slope s/d. Now show that

Lemma 3.2. *$E_2^{s,t}(G_j)$ and $E_2^{s,t}(R \otimes G_j)$ has a vanishing line with slope*

$$\frac{1}{2p^j n - 1}.$$

Assuming the lemma, choose j large enough so that

$$\frac{s}{d} < \frac{1}{2p^j n - 1}.$$

For any $\hat{\beta} \in E_2^{s,t}(R \otimes G_j)$, we have that $\hat{\beta}\hat{\alpha}^n = 0$ if $n \gg 0$. But this means $E_2^{s,t}(\alpha^{-1}R \otimes G_j) \equiv 0$. We conclude that

$$\pi_*(\alpha^{-1}R \otimes G_j) = 0.$$

There are different methods to establish Lemma 3.1. In [4] this is accomplished by finding a specific $X(n+1)$-Adams resolution for the spectra G_j and $R \otimes G_j$ whose kth associated graded is $(2p^j n - 1)s$-connected. An algebraic approach is taken in [1].

Remark 3.3. This part is where we need the assumption on connectivity of R (which turns out to be redundant).

4. Proof sketch for Step III

Below we sketch a proof of the following:

Theorem 4.1. $\langle G_j \rangle = \langle G_{j+1} \rangle$.

4.1. The self-map b.

As we will see, this eventually boils down to constructing a self-map

$$b : \Sigma^{2np^{j+1}-2} G_j \to G_j$$

and showing that it is nilpotent. We now skecth the construction of b. Take $p = 5$ for illustration purposes.

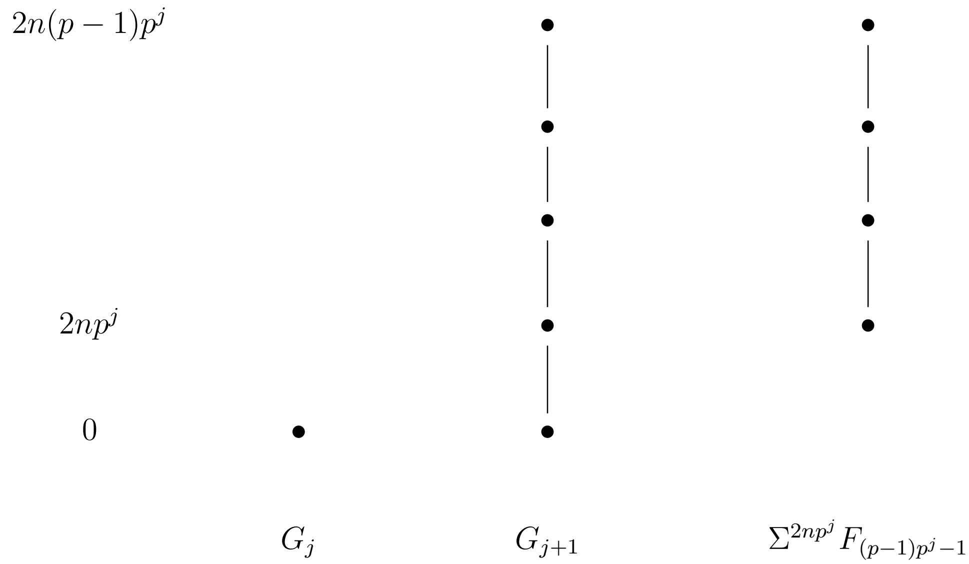
Recall that $G_j = F_{p^j - 1}$ and

$$\mathrm{H}_*(F_k) \cong \mathbf{Z}[b_1, b_2, \ldots, b_{n-1}]\{1, b_n, b_n^2, \ldots, b_n^k\},$$

We start from the cofiber sequence

$$G_j \to G_{j+1} \to \Sigma^{2np^j} F_{(p-1)p^j - 1}.$$

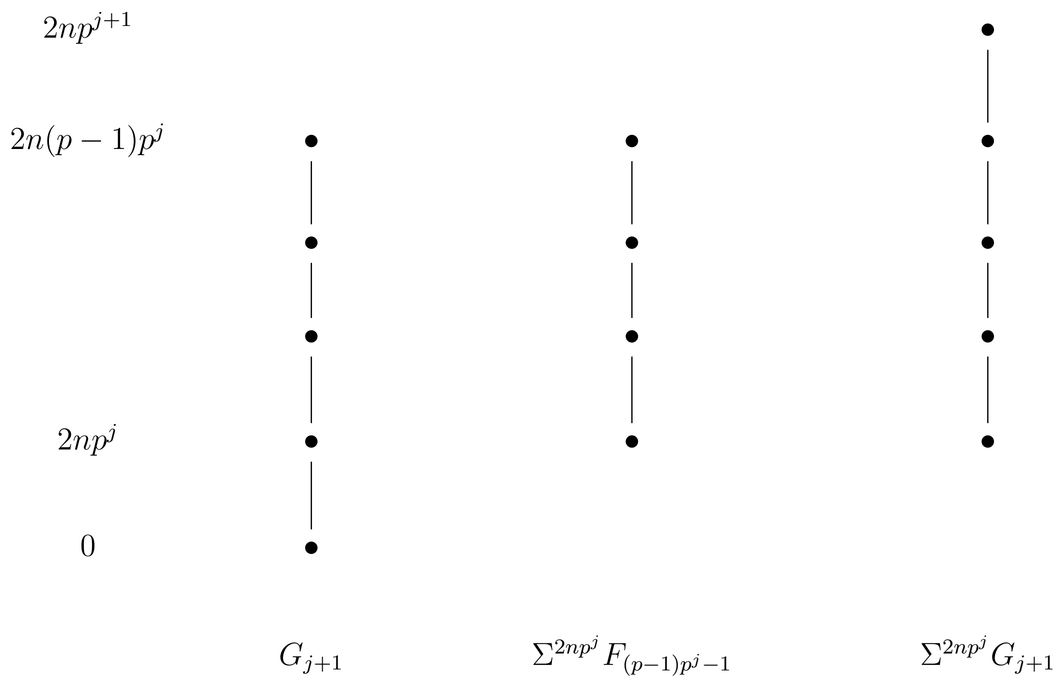
Represent G_j as a cell. The above cofiber sequence can be illustrated as follows:



Now consider the composition

$$r : G_{j+1} \to \Sigma^{2np^j} F_{(p-1)p^j-1} \to \Sigma^{2np^j} G_{j+1}$$

illustrated by



Define $K := \mathrm{cofib}(r)$. The spectrum $\Sigma^{-1}K$ has two "G_j cells", one in "dimension 0" and the other in "dimension $2np^{j+1}-1$".

Definition 4.2. Define the map $b : \Sigma^{2np^{j+1}-2} G_j \to G_j$ as the "attaching map" of the top cell of $\Sigma^{-1}K$. Namely, it fits into a cofiber sequence

$$\Sigma^{2np^{j+1}-2} G_j \xrightarrow{b} G_j \to \Sigma^{-1}K.$$

The following is immediate for dimension reasons:

Proposition 4.3. $(\mathrm{H}\mathbf{F}_p)_*(b) = 0$.

4.2. **Bousfield classes.** Recall the following result regarding Bousfield classes ([4, Proposition 7.2.6]):

Proposition 4.4. *(1) If $X \to Y \to Z$ is a cofiber sequence then $\langle Z\rangle \le \langle X\rangle \oplus \langle Y\rangle$*
(2) If $\Sigma^n X \xrightarrow{f} X \to C_f$ is a cofiber sequence then $\langle X\rangle = \langle C_f\rangle \oplus \langle f^{-1}X\rangle$.

Applying this proposition gives

(1) There is a cofiber sequence $G_{j+1} \xrightarrow{r} \Sigma^{2np^j} G_{j+1} \to K$. Therefore,
$$\langle K\rangle \le \langle G_{j+1}\rangle.$$
(2) There is a cofiber sequence $\Sigma^{2np^{j+1}-2} G_j \xrightarrow{b} G_j \to \Sigma^{-1}K$. Therefore,
$$\langle G_j\rangle = \langle b^{-1}G_j\rangle \oplus \langle K\rangle$$
(3) There's a filtration
$$G_j = F_{p^j-1} \hookrightarrow F_{2p^j-1} \hookrightarrow \cdots \hookrightarrow F_{p^{j+1}-1} = G_{j+1}$$
where all the successive cofibers are suspensions of G_j. Therefore,
$$\langle G_j\rangle \ge \langle G_{j+1}\rangle.$$

We conclude that
$$\langle G_{j+1}\rangle \le \langle G_j\rangle \le \langle G_{j+1}\rangle \oplus \langle b^{-1}G_j\rangle.$$
It now suffices to show $b^{-1}G_j \simeq *$. The proof of Theorem 4.1 thus reduces to the following claim:

Theorem 4.5. *The map $b : \Sigma^{2np^{j+1}} G_j \to G_j$ is nilpotent.*

4.3. **Nilpotence of b.** We will demonstrate nilpotence of b factoring it through Brown–Gitler spectra. The argument resembles Nishida's proof of Theorem 1.1, see [4, §9.6].

Recall that G_j is the Thom space of B_{p^j-1}, which is a pullback
$$\begin{array}{ccc} B_{p^j-1} & \longrightarrow & \Omega\mathrm{SU}(n+1) \\ \downarrow & & \downarrow \\ J_{p^j-1}S^{2n} & \longrightarrow & \Omega S^{2n+1} \end{array}$$
It turns out (Theorem A.3) that this can be extended to two fiber sequences
$$\begin{array}{ccccccc} \Omega^2 S^{2np^j+1} & \longrightarrow & B_{p^j-1} & \longrightarrow & \Omega\mathrm{SU}(n+1) & \longrightarrow & \Omega S^{2np^j+1} \\ \| & & \downarrow & & \downarrow & & \| \\ \Omega^2 S^{2np^j+1} & \longrightarrow & J_{p^j-1}S^{2n} & \longrightarrow & \Omega S^{2n+1} & \xrightarrow{H} & \Omega S^{2np^j+1} \end{array}$$
There is a "group action" that renders B_{p^j-1} as a "homogeneous space":
$$\Omega^2 S^{2np^j+1} \times B_{p^j-1} \to B_{p^j-1}.$$

Passing to Thom spectra one gets a map

$$\mu : \Sigma^\infty_+ \Omega^2 S^{2np^j+1} \otimes G_j \to G_j.$$

Recall the **Snaith splitting** (Theorem B.3 with $m = np^j$)

$$\Sigma^\infty_+ \Omega^2 S^{2np^j+1} \simeq (S^0 \oplus S^{2np^j-1}) \otimes \bigoplus_{i\geq 0} \Sigma^{i|b|} D_i$$

such that there are maps $\ell : D_i \to D_{i+1}$ with colimit $\mathrm{H}\mathbf{F}_p$ (see §B.2).

The following result is the key to the proof.

Proposition 4.6. *The map b^i factorizes as*

$$\begin{array}{ccc} \Sigma^{i|b|} D_i \otimes G_j & \longrightarrow & \Sigma^\infty_+ \Omega^2 S^{2np^j+1} \otimes G_j \\ \uparrow & & \downarrow \mu \\ \Sigma^{i|b|} G_j & \xrightarrow{\quad b^i \quad} & G_j \end{array}$$

From this we obtain a diagram

$$\begin{array}{ccccccc} G_j & \longrightarrow & D_1 \otimes G_j & \longrightarrow & D_2 \otimes G_j & \longrightarrow & \cdots \\ \| & & \downarrow \mu & & \downarrow \mu & & \\ G_j & \xrightarrow{b} & \Sigma^{-|b|} G_j & \xrightarrow{b} & \Sigma^{-2|b|} G_j & \longrightarrow & \cdots \end{array}$$

Taking colimits, this gives a factorization of $G_j \to b^{-1} G_j$ as

$$G_j \to \mathrm{H}\mathbf{F}_p \otimes G_j \to b^{-1} G_j$$

Inverting b in the above, one sees that $\mathrm{id}_{b^{-1}G_j}$ factors as

$$b^{-1} G_j \to \mathrm{H}\mathbf{F}_p \otimes b^{-1} G_j \to b^{-1} G_j,$$

but the middle map is trivial since $\mathrm{H}\mathbf{F}_p(b) = 0$ (Proposition 4.3). This means $b^{-1} G_j \simeq *$ as desired.

Appendix A. The James–Hopf map

The **James construction** on a pointed space X is given by

$$JX := \coprod_{i\geq 0} X^i / \sim,$$

where

$$(x_1, \ldots, x_j, *, x_{j+1}, \ldots, x_n) \sim (x_1, \ldots, x_j, x_{j+1}, \ldots, x_n).$$

In modern language, JX is an explicit model of the free $\mathbb{E}_1$-algebra on X.

The **kth stage** of the James construction on X is its k-skeleton

$$J_k(X) := \coprod_{k \geq i \geq 0} X^i / \sim$$

Theorem A.1. *We have*

$$JX \simeq \Omega \Sigma X, \quad \Sigma JX \simeq \bigvee_{i \geq 0} \Sigma X^{\wedge i}$$

The projection $\Sigma JX \to \Sigma X^{\wedge i}$ corresponds, by adjunction, to the **James–Hopf map**:

$$JX \xrightarrow{H} JX^i.$$

Remark A.2. (1) This is not a map of $\mathbb{E}_1$-algebras in general.
(2) This map is related to the Hopf invariant: take $X = S^n$ and $i = 2$. The map reads

$$\Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1},$$

which upon taking $\pi_{2n}(-)$ gives the Hopf invariant

$$\pi_{2n+1}(S^{n+1}) \to \pi_{2n+1}(S^{2n+1}) \cong \mathbf{Z}.$$

Specializing to the case where X is a sphere, one gets an explicit description of the fiber:

Theorem A.3. *We have a 2-local fiber sequence*

$$S^n \to JS^n \xrightarrow{H} JS^{2n}$$

and a p-local fiber sequence for $p > 2$

$$J_{p-1}S^{2n} \to JS^{2n} \xrightarrow{H} JS^{2np}.$$

Appendix B. The spectrum $\Sigma^\infty_+\Omega^2 S^{2m+1}$

Here we record some facts about $\Sigma^\infty_+\Omega^2 S^{2m+1}$, the free $\mathbb{E}_2$-algebra on S^{2m-1}. Most of the content here are contained in [4, §9.4].

B.1. **Snaith splitting.** We describe an explicit description of the splitting of $\Omega^2 S^{2m+1}$ after stabilization.

Proposition B.1. *For any $m > 0$,*

$$\mathrm{H}_*(\Omega^2 S^{2m+1}; \mathbf{F}_2) \cong \mathrm{P}(x_{2m-1}, x_{4m-1}, x_{8m-1}, \dots).$$

For any $m > 0$ and any $p > 2$,

$$\mathrm{H}_*(\Omega^2 S^{2m+1}; \mathbf{F}_p) \cong \mathrm{E}(x_{2m-1}, x_{2pm-1}, x_{2p^2m-1}, \dots) \otimes \mathrm{P}(y_{2pm-2}, y_{2p^2m-2}, \dots).$$

Here subscripts indicates the dimension of generators.

We assign a **weight** to each generator:

$$|x_{2p^im-1}| = |y_{2p^im-2}| = p^i.$$

Theorem B.2 (Snaith)**.** *There is a decomposition*

$$\Sigma^\infty_+\Omega^2 S^{2m+1} \simeq \bigoplus_{i\geq 0} D_{m,i},$$

with $\mathrm{H}_*(D_{m,i}; \mathbf{F}_p) \subset \mathrm{H}_*(\Omega^2 S^{2m+1}; \mathbf{F}_p)$ being the part spanned by monomials of weight i.

In particular, $D_{m,i} \simeq *$ unless $i \equiv 0, 1 \bmod p$. Furthermore, we have the following facts:

(1) $D_{m,0} \simeq S^0$.
(2) $D_{m,1} \simeq S^{2m-1}$.

(3) $D_{m,pi+1} \simeq \Sigma^{2m-1} D_{m,pi}$
(4) $D_{m,pi} \simeq \Sigma^{2i(pm-1)} D_i$, where the D_i's are **Brown–Gitler spectra** independent of m.
(5) $D_0 \simeq S^0$ and $D_1 \simeq S^0/p$.

The above splitting can thus be reformulated as follows.

Theorem B.3. *There is a decomposition*

$$\Sigma^\infty_+ \Omega^2 S^{2m+1} \simeq (S^0 \oplus S^{2m-1}) \otimes \bigoplus_{i \geq 0} \Sigma^{2i(pm-1)} D_i$$

B.2. **Construction of ℓ.** The map $\mu : \Omega^2 S^{2m+1} \times \Omega^2 S^{2m+1} \to \Omega^2 S^{2m+1}$ stabilizes to give multiplications

$$D_{m,i} \otimes D_{m,j} \to D_{m,i+j}.$$

In particular, one gets a map

$$\ell : D_i \to D_1 \otimes D_i \to D_{i+1}.$$

The effect of this on homology is multiplicaiton by y_{2pm-2}. Thus,

$$\mathrm{H}_*(\mathrm{colim}_{\,\ell}(D_i)) \subset y^{-1}_{2pm-2} \mathrm{H}_*(\Sigma^\infty_+ \Omega^2 S^{2m+1})$$

can be identified with the weight 0 part of the right hand side. This is isomorphic to $\mathrm{H}\mathbf{F}_{p*}\mathrm{H}\mathbf{F}_p$ as a right $\mathcal{A}$-module, so we have

$$\mathrm{colim}_{\,\ell}(D_i) \simeq \mathrm{H}\mathbf{F}_p.$$

References

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