PROVING BOTT PERIODICITY FOR HOMOTOPY THEORISTS

<u>REFERENCES:</u>

- Atiyah-Singer, "Index theory for skew-adjoint Fredholm operators."
- McDuff, "Configuration spaces".
- * Behrens, "A new proof of Bott periodicity" & its <u>Addendum</u>

$$\operatorname*{colim}_n O(n) =: O \simeq \Omega(\Omega^\infty KO)$$

$$\operatorname*{colim}_n U(n) =: U \simeq \Omega(\Omega^\infty KU)$$

$$\left(\operatorname*{colim}_n \mathrm{Iso}(\mathbb{H}^n) =: Sp \simeq \Omega(\Omega^\infty KSp)\right)$$

Theorem: (Bott, 1958)

- $\pi_k(U) \cong \pi_{k+2}(U)$
- $\pi_k(O) \cong \pi_{k+4}(Sp)$
- $\pi_k(Sp) \cong \pi_{k+4}(O)$

$$\left.\begin{array}{l} \pi_k(O) \cong \pi_{k+4}(Sp) \\ \pi_k(Sp) \cong \pi_{k+4}(O) \end{array}\right\} \pi_k(O) \cong \pi_{k+8}(O).$$

This full theorem is "Bott periodicity"

Note: $\pi_k(\Omega^\ell X) \cong \pi_{k+\ell}(X)$

$\Rightarrow$ Can rephrase in terms of loop spaces.

Bott periodicity

- $U \simeq \Omega^2 U$
- $O \simeq \Omega^8 O$

Bott's proof: Spaces of minimal geodesics & Morse theory.

Note: O & U are defined in the same way but over different fields $\mathbb{R}$ & $\mathbb{C}$. Similarly for KO & KU. $\Rightarrow$ Look for proofs in this spirit that work over $\mathbb{R}$ & $\mathbb{C}$ simultaneously.

Atiyah-Bott-Shapiro (1963)

Clifford algebra

$$\mathcal{M}_k^{\mathbb{C}} = \mathbb{Z}\{\text{irrep's of } \mathbb{C}\ell_k\} \to \mathcal{M}_{k-1}^{\mathbb{C}}$$

$$\mathcal{M}_k^{\mathbb{C}} / \mathcal{M}_{k+1}^{\mathbb{C}} \cong \pi_k(KO) \cong \pi_{k-1}(O)$$

$$\cong \pi_k(KU) \cong \pi_{k-1}(U)$$

<u>$k=0$</u>: • $\mathbb{C}\ell_0 = \mathbb{R}$ or $\mathbb{C}$

• KO / KU = vector bundles

<u>Insight</u>: $\mathbb{C}\ell_n$ is the higher degree analog of the ground field.

Look for proof of Bott periodicity in these terms.

Atiyah-Singer (1969)

Realize above goal w/ help from analysis.

$V \xrightarrow{T} W \implies$ "dimension theorem"

$$\dim V = \operatorname{rank} T + \operatorname{nullity} T$$

$$\operatorname{rank} T = \dim W - \dim(\operatorname{coker} T)$$

$$\underbrace{\dim V - \dim W}_{\text{"K-theory"}} = \underbrace{\dim(\ker T) - \dim(\operatorname{coker} T)}_{\text{Index } T}$$

"K-theory" $\downarrow$ finite dim's

Index T $\downarrow$ infinite dim's too

if T is Fredholm.

virtual bundles $\longrightarrow$ families of Fredholm operators

degree $k > 0$ $\longrightarrow$ $C\ell_k$ replaces $\mathbb{R}$ / $\mathbb{C}$.

$\mathcal{H}_n$ = universal (Hilbert) $C\ell_n$-module

Fred_n = "space of skew-adjoint $C\ell_n$-linear Fredholm operators on $\mathcal{H}_n$". (modulo some technical details)

$\downarrow$

represents KO^n/KU^n $\rightarrow$ where the analysis lies.

periodicity: $C\ell_n \underset{\text{Morita}}{\simeq} C\ell_{n+8}^{\;+2}$ $\Longrightarrow$ periodicity of modules.

McDuff (1975)

fibration $\begin{array}{ccc} F & \to & E \\ & & \downarrow p \\ & & X \end{array}$ $\Rightarrow$ long exact sequence of homotopy groups.

Actually, sufficient for p to be a quasifibration

$:=$ surjective & $p_* : \pi_i(E, F_x, x) \xrightarrow{\sim} \pi_i(X, x), \forall i.$

contractible

quasifib.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & X \arrow[d, equal] \\ \Omega X \arrow[r] & PX \arrow[r] & X \end{tikzcd}$$

long exact sequence & 5-lemma

$\Downarrow$

$F \xrightarrow{\sim} \Omega X$

⇝ General method for showing one space is a loop space of another.

McDuff's sketch of the complex case

① We already have $U \to EU \to BU$

$\Rightarrow U \simeq \Omega BU \simeq \Omega(BU \times \mathbb{Z})$.

② Want $BU \times \mathbb{Z} \to \overset{*}{\underset{12}{E}} \xrightarrow{\text{quasifib.}} U$

Her idea: use the Atiyah-Singer idea but replace "Hilbert" ⇝ colim ("finite dim")

$\Rightarrow$ Contact w/ $\mathcal{U}$ easier, & hides the work in verifying the quasifib. property.

$$\begin{array}{l} E(k) = \text{skew-adjoint matrices w/ eigenvalues} \in [0,1]. \\ \downarrow \exp(k) \\ \mathcal{U}(k) \end{array}$$

$\rightsquigarrow$ show $B\mathcal{U} \times \mathbb{Z}$ is the fiber of $\operatorname{colim}_k (\exp(k))$

$\hookrightarrow$ analogous to the proof that $\Omega S^1 \simeq \mathbb{Z}$

But we want to handle real & complex cases with the same method.

McDuff suggests that this can be done using the Atiyah-Singer idea, but she doesn't pursue this.

Behrens (2002 - 2004)

Plan:

(I will use only $\mathbb{R}$ - notation. Everything works for $\mathbb{C}$ as well)

1. Define $X(n)$ & $E(n)$ (a la Atiyah-Singer)
2. $E(n) \xrightarrow{p} X(n)$ quasifibration
   Dold-Thom for detecting quasifib's
3. $fib(p) \simeq X(n+1)$
4. $E(n)$ contractible
5. $X(n+8) \simeq X(n)$
6. $X(1) \simeq O/U$

trivial

We'll do the constructions & show these

Definitions

Let W be a Cl_{n-1}-module. (W has $\langle , \rangle$ & Cl_{n-1} acts by isometries)

$$X(n,W) := \left\{ \begin{array}{c} Cl_n\text{-extensions} \\ \text{of } W \end{array} \right\} \subseteq O(W)$$

notation e_n^ϵ

Let $\mathcal{U}$ be a <u>complete Cl_n-universe</u> ($\sim \mathcal{H}_n$)

$$X(n) := \operatorname*{colim}_W X(n,W)$$

Given $W \subseteq \mathcal{U}$, $E(n,W) := \left\{ \begin{array}{l} Cl_{n-1}\text{-linear} \\ \text{skew-adjoint } A \\ \text{on } W \text{ w/ e-val's} \\ \subseteq [-i, i] \text{ \& } e_n A = -A e_n \end{array} \right\} \subseteq \mathfrak{o}(W)$

$$E(n) := \operatorname*{colim}_W E(n,W).$$

$$E(n,W) \xrightarrow{P_W} X(n,W)$$

$$A \longmapsto \exp(A)\, e_n \exp(A)^{-1}$$

$$P := \operatorname*{colim}_W (P_W) : E(n) \longrightarrow X(n).$$

The easy parts

4 by definition (convex $\subseteq$ vector space)

5 $Cl_n \underset{\text{Morita}}{\simeq} Cl_{n+8}$

6 $X(1, W) =$ extensions of a $\mathbb{R}$- inner product space $\longrightarrow$ Cl_0-module W to a Cl_1-module $\uparrow$ $\mathbb{C}$- inner product space

i.e.
$$X(1) \simeq \mathrm{fib}(BU \to BO)$$
$$\simeq \Omega\, \mathrm{cofib}(BU \to BO)$$
$$\simeq \Omega B(O/U)$$
$$\simeq O/U. \quad //$$