BP-theory & the Adams spectral sequence

References:

- Ravenel, "Nilpotence & periodicity in stable homotopy theory."
- Ravenel, "Complex cobordism & stable homotopy groups of spheres."
- Wilson, "Brown-Peterson homology: An introduction and sampler."

The original homology theory $H\mathbb{Z}$

idea: probe a space X with the most basic n-dim'l objects.

$$\sigma : \Delta^n \longrightarrow X$$

$$\Rightarrow \quad S_* X \quad \Rightarrow \quad H_*(X; \mathbb{Z}).$$

- degree n homology classes represent n-dim'l "shapes" present in X.

A model for n-dim'l shapes: Manifolds

manifold $M \implies [M] \in H_*(M; \mathbb{Z}/2)$

$f: M \xrightarrow{\text{space}} X \implies f_*[M] \in H_*(X; \mathbb{Z}/2)$

Thm: (Thom) $\forall X, \forall \alpha \in H_*(X; \mathbb{Z}/2)$

$\exists f: M \to X$ s.t. $\alpha = f_*[M]$.

We can rephrase this as:

$$\boxed{MO \simeq \bigvee_k H\mathbb{Z}/2}$$

What about $H_*(X;\mathbb{Z})$?

oriented mfd $M \Rightarrow [M] \in H_*(M;\mathbb{Z})$

$f: M \longrightarrow X \Rightarrow f_*[M] \in H_*(X;\mathbb{Z})$

⚠ NOT EVERY HOMOLOGY CLASS ARISES IN THIS WAY!

why? MSO is not a wedge of Eilenberg-Mac Lane spectra!

(even though $MSO_{(2)} \simeq (\bigvee_i H\mathbb{Z}) \vee (\bigvee_j H\mathbb{Z}/2)$)

Enter the Brown-Peterson spectrum

Eilenberg-Mac Lane spectra are not sufficient to understand cobordism theories.

Thm: (B-P, 1966) $\forall$ prime p, $\exists$ spectrum BP s.t. $MU_{(p)} \simeq \bigvee_k \Sigma^{n_k} BP$.

p-localization (future talk)

Furthermore, when localized at odd primes, MSO, MSU, & MSp are also wedges of suspensions of BP. (think "BP = prime bordism")

Some properties of BP

- $\pi_* BP \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, $|v_i| = 2p^i - 2$
- $H_* BP \cong \mathbb{Z}_{(p)}[t_1, t_2, \ldots]$, $|t_i| = 2p^i - 2$
- Hurewicz: $\pi_* BP \longrightarrow H_* BP$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v_i & \longmapsto & p t_i + \text{decomposables} \end{array}$$

$\hookrightarrow$ $\pi_* BP \subseteq H_*(BP)$

Looks similar to MU, but much smaller: degrees of generators grow exponentially, not linearly.

(for MU, $|x_i| = 2i$)

Recall: Quillen's theorem $L \xrightarrow{\sim} \pi_* MU$

$\Rightarrow$ the formal group law associated to MU is the universal one.

Def: A formal group law F over a $\mathbb{Z}_{(p)}$-algebra is <u>p-typical</u> if

$$f_q(x) = 0 \quad \forall \text{ primes } q \neq p,$$

$$\frac{1}{q} \sum_{i=1}^{q}{}_F \, \zeta^i x \,, \quad \zeta = \text{primitive } q^{th} \text{ root of } 1.$$

Over a torsion-free $\mathbb{Z}_{(p)}$-algebra,

$$\left(F \text{ p-typical}\right) \iff \left(\log_F(x) = \sum_{i \geq 0} l_i x^{p^i},\ l_0 = 1\right).$$

Thm: (Quillen) The formal group law associated to BP is the universal p-typical fgl over $\pi_* BP$.

It's induced by a homomorphism

$$MU_* \otimes \mathbb{Z}_{(p)} \longrightarrow BP_* .$$

The Hopf algebroid $BP_*(BP)$

- ring: $BP_*(BP) \cong BP_*[t_1, t_2, \dots]$, $|t_i| = 2p^i - 2$.
- coproduct: $\sum_{i,j\geq 0} l_i \Delta(t_j)^{p^i} = \sum_{i,j,k\geq 0} l_i t_j^{p^i} \otimes t_k^{p^{i+j}}$.
  $(t_0 = 1)$
- left unit: $\eta_L : BP_* \hookrightarrow BP_*(BP)$ inclusion.
- right unit: $\sum_{i\geq 0} \eta_R(l_i) = \sum_{i,j\geq 0} l_i t_j^{p^i}$

In contrast to MU, the Hopf algebroid $BP_*(BP)$ cannot be constructed from any Hopf algebra over $\mathbb{Z}_{(p)}$.

A perspective of Wilson & Priddy (1980)

[Wilson]	$S_{(p)} \longrightarrow BP \longrightarrow H\mathbb{Z}_{(p)}$		
homology	good	okay	bad
homotopy	bad	okay	good

[Priddy] $\quad S_{(p)} \xrightarrow{\text{kill most homotopy gps}} H\mathbb{Z}_{(p)}$

via attaching cells

constructs $\quad S_{(p)} \xrightarrow[\text{kill odd homotopy groups}]{} BP$

Ravenel: In practice, BP computations are hard

$\Longrightarrow$ usually compute modulo an ideal.

Thm: (Morava, Landweber)

- $I_n := (p, v_1, \ldots, v_{n-1}) \subseteq BP_*$ is an invariant prime ideal.
- These are the only invariant prime ideals in BP_*.

Smaller versions of BP

Sullivan-Baas: construct a spectrum $C(y_1, \dots, y_{n-1})$ with $\pi_* C(y_1, \dots, y_{n-1}) \cong \pi_* MU/(y_1, \dots, y_{n-1})$.

Johnson-Wilson: apply Sullivan-Baas to get

$$BP\langle n \rangle \quad \text{with} \quad \pi_* BP\langle n \rangle = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$$

Ex: $BP\langle 1 \rangle$ is a direct summand of $ku_{(p)}$.

Prop: X = finite CW complex $\Rightarrow BP_*(X)$ can be computed using $BP\langle n \rangle_*(X)$ for some n.

The Adams spectral sequence

Given a homology theory E_*,
want a spectral sequence $\{E_r^{*,*}\}_r \to \pi_*(X)$
(at least at a prime p) whose E_2-page
is a functor of $E^*(X)$ as a $E^*(E)$-module
-OR- " of $E_*(X)$ as a $E_*(E)$-comodule.

Ex: For $E = H\mathbb{Z}/p$,

$$E_2 = \mathrm{Ext}_{A_*}(\mathbb{Z}/p, H_*(X))$$

Def: The <u>canonical Adams resolution</u> for X based on E is the diagram

$$\begin{tikzcd}
X = X_0 \arrow[d, "f_0"'] & X_1 \arrow[l] \arrow[d, "f_1"'] & X_2 \arrow[l] \arrow[d, "f_2"'] & \cdots \arrow[l] \\
E \wedge X_0 & E \wedge X_1 & E \wedge X_2 &
\end{tikzcd}$$

induced by the unit of E.

where $X_{n+1} = \mathrm{fib}(f_n)$.

Each $X_{n+1} \to X_n \to E \wedge X_n \implies$ long exact sequence of homotopy groups.

Put these together into an exact couple
$\Rightarrow$ the <u>Adams spectral sequence</u> for X based on E.

<u>Rmk</u>: other "Adams resolutions" also yield s.s.'s.

<u>Bousfield</u>: gives conditions for convergence.

<u>The E_2-page</u>

when E is <u>flat</u>

$$\Rightarrow \quad E_*(E \wedge X) \cong E_*(E) \otimes_{\pi_* E} E_*(X).$$

As with $H\mathbb{Z}/p$ above, the E_2-page based on a flat ring spectrum can be described as an Ext group in the category of $E_*(E)$-comodules.

Like MU, BP is flat &

<u>Thm:</u> (Novikov) The BP-based Adams s.s. has

$$E_2 \cong \mathrm{Ext}_{BP_*(BP)}\left(BP_*, BP_*(X)\right)$$

called the <u>Adams-Novikov spectral sequence</u>.