Vector Bundles and Homogeneous Spaces

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Top = category of spaces that admit a finite CW structure.

Top_* = cat. of based spaces $\in$ Top

Top^2 = cat. of pairs (X,A) in Top.

Generalized cohomology theory

- a functor $E^\bullet : Top^2 \longrightarrow Ab_{\mathbb{Z}}$ ← $\mathbb{Z}$-graded
- a natural transformation of degree $+1$

$$\delta : E^\bullet(A,\emptyset) \longrightarrow E^{\bullet+1}(X,A)$$

satisfying

- (homotopy) $f_1, f_2 : (X,A) \longrightarrow (Y,B)$ homotopic $\Rightarrow E^\bullet(f_1) = E^\bullet(f_2)$
- (exactness) $(X,A) \in Top^2 \Rightarrow$ L.E.S.

$$\cdots \longrightarrow E^n(X,A) \to E^n(X,\emptyset) \to E^n(A,\emptyset) \xrightarrow{\delta} E^{n+1}(X,A) \to \cdots$$

- (excision) $(X,A) \in Top^2$, $\bar{U} \subseteq \operatorname{Int} A$

$$\Rightarrow E^\bullet(i) : E^\bullet(X,A) \xrightarrow{\sim} E^\bullet(X \setminus U, A \setminus U)$$

- (additivity) $E^n(\coprod_i X_i, \coprod_i A_i) \xrightarrow{\sim} \prod_i E^n(X_i, A_i)$

Notation

- $E^\bullet(X) = E^\bullet(X, \emptyset)$
- $\tilde{E}^\bullet(X) = E^\bullet(X, pt)$

For any cohomology theory we have

- Mayer-Vietoris sequences, &
- suspension isomorphism:

$$\tilde{E}^n(X) \cong \tilde{E}^{n+1}(\Sigma X)$$

$\Rightarrow$ knowing $\tilde{E}^n$ determines $\tilde{E}^k$ $\forall k \leq n$.

i.e. $\tilde{E}^k(X) \cong \tilde{E}^n(\Sigma^{n-k} X)$

Brown: $\tilde{E}^n(X) \cong [X, B_n]$

susp $\Rightarrow$ $\tilde{E}^n(X) \cong \tilde{E}^{n+1}(\Sigma X)$

$$\cong [\Sigma X, B_{n+1}]$$

$$\cong [X, \Omega B_{n+1}]$$

$\Rightarrow$ $\tilde{E}^\bullet$ rep'd by $\{B_n\}_{n\in\mathbb{Z}}$, with $B_n \simeq \Omega B_{n+1}$.

Singular cohomology: $\Delta^n \longrightarrow X$

<u>K-theory:</u> vector bundles $V \to X$.

can have different ranks on different components

$$\mathrm{Vect}(X) = \{\mathbb{C}\text{-vector bundles on } X\}/\text{isomorphism}$$

↳ monoid under direct sum $\oplus$

$$K(X) := \mathbb{Z}\,\mathrm{Vect}(X) \big/ V+W \sim V \oplus W$$

$$= \{[V]-[W] \mid V, W \in \mathrm{Vect}(X)\}$$

trivial bundle of rank n

$$= \{\underbrace{[V] - \underline{n}}_{\text{this representation is not unique!}} \mid V \in \mathrm{Vect}(X),\ n \in \mathbb{Z}_{\geq 0}\}$$

<u>Def:</u> $V \in \mathrm{Vect}(X)$ is <u>irreducible</u> if $V \not\cong W \oplus \underline{n}$ for any $W \in \mathrm{Vect}(X)$, $n \in \mathbb{Z}_{\geq 0}$.

even with irred. could have $[V] + \underline{n} = [W] + \underline{n}$.

<u>Def:</u> $V, W \in \mathrm{Vect}(X)$ are <u>stably equivalent</u> if $\exists n \in \mathbb{Z}_{\geq 0}$ s.t. $V \oplus \underline{n} \cong W \oplus \underline{n}$.

$$V \underset{st}{\sim} W \implies [V] = [W] \in K(X).$$

Then

$$K(X) = \left\{ \underbrace{[V] + \underline{n}}_{\text{This representation } \underline{\text{is}} \text{ unique}} \;\middle|\; \begin{array}{l} V \text{ stable equiv. class of irred.} \\ \text{vector bundles}, \quad n \in \mathbb{Z} \end{array} \right\}$$

$$\cong BU^X \times \mathbb{Z}$$

<u>Recall</u>: rank n vector bundles are represented by the Grassmannian $Gr_n(\mathbb{C}^\infty) = \operatorname{colim}_k Gr_n(\mathbb{C}^k)$

$$\mathrm{Vect}_n(X) \cong [X, \underset{\displaystyle BU(n)}{\overset{Gr_n(\mathbb{C}^\infty)}{\|}}]$$

$$BU(n) \longrightarrow BU(n+m) \quad \text{induced by } \oplus \underline{m}$$

$$\Rightarrow BU := \operatorname{colim}_n BU(n)$$

↑ this procedure corresponds to "irred" & stability above

$$\Rightarrow [X, BU] \cong \{\text{stable equiv. classes of irred. v.b.'s}\}$$

$$\Rightarrow K(X) \cong [X, BU \times \mathbb{Z}] \Rightarrow BU \times \mathbb{Z} \text{ represents } K.$$

to get a cohomology theory,

$$\Rightarrow K^{-n}(X) \cong [X, \Omega^n(BU \times \mathbb{Z})]$$

<u>Observe:</u> BU is a classifying space for

$U = \operatorname{colim}_n U(n) \implies U \simeq \Omega BU \simeq \Omega(BU \times \mathbb{Z})$.

<u>Thm:</u> (Bott periodicity)

$$\Omega U \simeq BU \times \mathbb{Z}$$

$\implies$ the sequence $\{\Omega^n(BU \times \mathbb{Z})\}_{n \geq 0}$ is 2-periodic!

$\implies$ extend the definition of KO^n to $n > 0$.

<u>Def:</u> $X \in \mathrm{Top}$, $n \in \mathbb{Z}$.

$$K^n(X) = \begin{cases} [X, BU \times \mathbb{Z}], & \text{even} \\ [X, U], & n \text{ odd} \end{cases}$$

<u>$K^\bullet$ in terms of vector bundles & suspension</u>

- $$\tilde{K}(X) = \ker\Big(K(X) \longrightarrow K(pt) \cong \mathbb{Z}\Big)$$
$$[V]-[W] \longmapsto \operatorname{rank} V - \operatorname{rank} W$$
$$= \{[V] - \operatorname{rank} V \mid V \text{ stble irred.}\}$$
$$\cong [X, BU].$$

- $K(X,Y) = \tilde{K}(X/Y)$ $\left(\Rightarrow \tilde{K}(X) = K(X, pt)\right)$

  letting $X/_{\emptyset} = X \amalg \{*\} \Rightarrow K(X) = K(X, \emptyset)$

- $K^n(X,Y) = \begin{cases} K(X,Y) & , n \text{ even} \\ \tilde{K}(\Sigma(X/Y)), & n \text{ odd} \end{cases}$

$\Rightarrow \forall n \in \mathbb{Z}: \quad K^n(X,Y) \cong K^{n+2}(X,Y).$

[**Thm:** $K^{\bullet}$ is a cohomology theory

exactness: $Y \xrightarrow{f} X \to C_f \longrightarrow \Sigma Y \xrightarrow{\Sigma f} \Sigma X \to \cdots$

$[-, B] \Rightarrow$ exact sequence

$\cdots \to K^{n+1}(X) \to K^{n+1}(Y) \to K^n(X,Y) \to K^n(X) \to \cdots$

The multiplicative structure

$K(X)$ inherits a ring structure from the semiring $(\mathrm{Vect}(X), \oplus, \otimes)$.

For other n: tensor product induces

$$K(X) \otimes K(Y) \longrightarrow K(X \times Y)$$
$$\tilde{K}(X) \otimes \tilde{K}(Y) \longrightarrow \tilde{K}(X \wedge Y)$$

more generally,

$$K(X, X_0) \otimes K(Y, Y_0) \longrightarrow K(X \times Y, X_0 \times Y \cup X \times Y_0).$$

using

$$\Sigma^m(X/X_0) \wedge \Sigma^n(Y/Y_0) = \Sigma^{m+n}(X \times Y / X_0 \times Y \cup X \times Y_0)$$

$$\Rightarrow K^{-n}(Y, Y_0) \times K^{-m}(X, X_0) \longrightarrow K^{-(n+m)}(X \times Y, X_0 \times Y \cup X \times Y_0)$$

Prop:

- $\bigoplus_{n \geq 0} K^{-n}(X)$ is a graded-commutative ring
- $\bigoplus_{m \geq 0} K^{-m}(X, Y)$ is a graded module over $\oplus K^{-n}(X)$

↳ most of this is repetitive → describe Bott periodicity in terms of the ring structure to make these rings/modules smaller.

The Bott class

Recall: The space $K(\mathbb{Z},2)$ representing $H^2(-;\mathbb{Z})$ is determined by $\pi_n K(\mathbb{Z},2) = \begin{cases} \mathbb{Z}, & n=2 \\ 0, & \text{else} \end{cases}$

$$\pi_n(U(1)) = \pi_n(S^1) = \begin{cases} \mathbb{Z}, & n=1 \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow \quad \pi_n(BU(1)) = \pi_n(K(\mathbb{Z},2)) \quad \forall n$$

$$\Rightarrow \quad \mathbb{CP}^\infty = BU(1) \simeq K(\mathbb{Z},2)$$

$$\Rightarrow \mathrm{Vect}_1(X) \xrightarrow[c_1]{\sim} H^2(X;\mathbb{Z})$$

c_1 = the first Chern class.

$$H^2(S^2;\mathbb{Z}) \cong \mathbb{Z} \Rightarrow \eta \in \mathrm{Vect}_1(X) \text{ with } c_1\eta = 1.$$

Thm: Let $\beta = [\eta] - \underline{1} \in \tilde{K}(S^2) = K^{-2}(pt)$.

$\beta \cdot (-) : K^{-n}(X, X_0) \xrightarrow{\sim} K^{-n-2}(X, X_0)$

is an isomorphism.

$$\Rightarrow \bigoplus_{n \geq 0} K^{-n}(x_0) \cong \mathbb{Z}[\beta] \quad \text{as rings.}$$

$\Rightarrow$ We can define $K^*(X,Y) = K^0(X,Y) \oplus K^1(X,Y)$ which is a $\mathbb{Z}/2$-graded module over the $\mathbb{Z}/2$-graded ring $K^*(X) = K^0(X) \oplus K^1(X)$.

$\Rightarrow$ exact sequence

$$\begin{tikzcd}
K^*(Y) \arrow[rr] & & K^*(X,Y) \arrow[dl] \\
& K^*(X) \arrow[ul] &
\end{tikzcd}$$

or

$$\begin{tikzcd}
& K^1(X,Y) \arrow[r] & K^1(X) \arrow[dr] & \\
K^0(Y) \arrow[ur] & & & K^1(Y) \arrow[dl] \\
& K^0(X) \arrow[ul] & K^0(X,Y) \arrow[l] &
\end{tikzcd}$$

Ex: $\tilde{K}^*(S^n) = \begin{cases} \mathbb{Z}, & * = n \mod 2 \\ 0, & \text{else} \end{cases}$

$\cong \tilde{H}^*(S^n; \mathbb{Z})$

The relation between K^* & H^*

Chern classes of a vector bundle

$$H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \dots], \quad |c_k| = 2k.$$

Given a v.b. $X \xrightarrow{V} BU(n)$

define $c_k V = V^* c_k \in H^{2k}(X;\mathbb{Z})$

total chern class: $c = \sum_k c_k$

$$\Rightarrow \quad c(E \oplus E') = c(E)\, c(E').$$

[Splitting principle: Any v.b. can by pulled back to a sum of line bundles $\Rightarrow$ knowing c_1 is enough.]

Chern character

$$\mathrm{ch}: \mathrm{Vect}(X) \longrightarrow H^{ev}(X;\mathbb{Q})$$

$$\bigoplus_{i=1}^{n} L_i \longmapsto \sum_{i=1}^{n} e^{c_1(L_i)}$$

& extend by naturality.

$$\Rightarrow \quad \mathrm{ch}: K(X) \longrightarrow H^{ev}(X;\mathbb{Q})$$

In fact, can extend to natural

$$ch : K^*(X,Y) \longrightarrow H^*(X,Y;\mathbb{Q})$$

preserving the $\mathbb{Z}/2$-grading.

Thm:

(1) $ch : K^*(X)\otimes\mathbb{Q} \xrightarrow{\sim} H^*(X;\mathbb{Q})$

(2) If $K^*(X)$ has no torsion, then

$ch : K^*(X) \longrightarrow H^*(X;\mathbb{Q})$ is injective.

$\Rightarrow$ The main difference btwn K^* & H^* concerns torsion. (torsion is where the difficulty of algebraic topology lies anyway)

How to prove the thm?

↳ <u>Atiyah-Hirzebruch spectral sequence</u>.

Let X = finite simplicial cpx $\Rightarrow X = \bigcup_p X^p$

$\Rightarrow$ filtration $K^n_p(X) = \ker\left(K^n(X) \to K^n(X^{p-1})\right)$

Thm: $\exists$ spectral sequence $\{E_r^{p,q}\} \longrightarrow G_p K^{p+q}(X)$ with $E_2^{p,q} \cong H^p(X; K^q(x_0))$.

Note: $K^q(x_0) = 0$ for q odd

$\Rightarrow$ only get nonzero rows at even heights.

Ex: $X = \Sigma^g$ orientable surface of genus g.

$\boxed{E_2}$

2	$\mathbb{Z}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}$
1	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}$
	0	1	2

zeros $\Rightarrow$ d vanishes $\Rightarrow$ $K^0(\Sigma^g) \cong \mathbb{Z}^2$

$K^1(\Sigma^g) \cong \mathbb{Z}^{2g}$.

Ex: $H^*(\mathbb{RP}^{2n}; \mathbb{Z}) \cong \mathbb{Z} \oplus \underbrace{\mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2}_{\text{even dim's.}}$

$$\begin{aligned} \mathrm{Vect}_1(\mathbb{RP}^{2n}) &\cong H^2(\mathbb{RP}^{2n}) \cong \mathbb{Z}/2 \\ L &\longmapsto 1 \end{aligned}$$

$\Rightarrow [L] - \underline{1} \in \tilde{K}(\mathbb{RP}^{2n})$

Can show $2([L] - 1) \neq 0$

$\Rightarrow \tilde{K}^*(\mathbb{RP}^{2n}) \neq \tilde{H}^*(\mathbb{RP}^{2n}; \mathbb{Z})$