The (strict) 2-category of categories

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The goal of this document is to show that categories, functors, and natural transformations form a strict 2-category. In particular, I (try to) write out all of the necessary details, using diagrams over equations whenever relevant. I start by giving a definition of strict 2-category for the sake of referencing during the proof. I then define the data of the 2-category of categories and show that they satisfy the definition of a (strict) 2-category.

Definition: A **strict 2-category** C consists of

1. a collection of **0-cells**;
2. for each pair of 0-cells (x, y), a category $C(x, y)$, whose objects are called **1-cells** and whose morphisms are called **2-cells**; composition of 2-cells within the categories $C(x, y)$ is called **vertical composition**;
3. for each 0-cell x, a distinguished 1-cell $1_x \in \operatorname{ob} C(x, x)$ called the **identity 1-cell** at x;
4. for each triple of 0-cells (x, y, z), a functor $\circ : C(y, z) \times C(x, y) \to C(x, z)$ called **horizontal composition**;

which satisfy the following conditions:

I. (identities)

- the identity 1-cells are strict left and right units with respect to horizontal composition, i.e. for each 1-cell $f \in \operatorname{ob} C(x, y)$,
$$f \circ 1_x = f \quad \text{and} \quad 1_y \circ f = f,$$
- the identity 2-cells corresponding to the identity 1-cells are strict left and right units with respect to horizontal composition, i.e. for each 2-cell $\alpha \in \operatorname{mor} C(x, y)$,
$$\alpha \circ 1_{1_x} = \alpha \quad \text{and} \quad 1_{1_y} \circ \alpha = \alpha;$$

II. (associativity) horizontal composition is strictly associative, i.e.

- for each triple of 1-cells $(h, g, f) \in \operatorname{ob} C(y, z) \times \operatorname{ob} C(x, y) \times \operatorname{ob} C(w, x)$,
$$h \circ (g \circ f) = (h \circ g) \circ f,$$
- for each triple of 2-cells $(\gamma, \beta, \alpha) \in \operatorname{mor} C(y, z) \times \operatorname{mor} C(x, y) \times \operatorname{mor} C(w, x)$,
$$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha.$$

Let's verify that if we define

- 0-cells = categories,
- 1-cells = functors,
- 2-cells = natural transformations,

then we get a strict 2-category.

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1. Given.
2. For two categories C, D, we already know that the functors from C to D together with the natural transformations between such functors form a category $\mathsf{Fun}(C, D)$. Here are the main observations:
   - The identity natural transformation 1_F is defined as the identity map at every object c; $1_F(c) = 1_{F(c)}$.
   - Composition is defined "component-wise" at each object, so associativity is thus inherited from associativity of D.
3. For a category C, the identity functor $1_C : C \to C$ is the distinguished identity 1-cell.
4. Given categories C, D, E, we need a composition functor

$$\circ : \mathsf{Fun}(D, E) \times \mathsf{Fun}(C, D) \to \mathsf{Fun}(C, E).$$

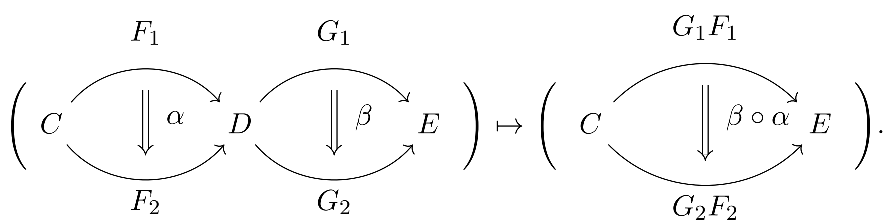
On objects (i.e. 1-cells), this is simply composition of functors:

$$(\ C \xrightarrow{F} D \xrightarrow{G} E\ ) \mapsto (\ C \xrightarrow{G \circ F} E\ ).$$

For a pair of morphisms (= 2-cells = natural transformations) $(\beta, \alpha) \in \mathsf{Fun}(D, E) \times \mathsf{Fun}(C, D)$, where $\alpha : F_1 \Rightarrow F_2 : C \to D$ and $\beta : G_1 \Rightarrow G_2 : D \to E$, we need to define a natural transformation

$$\beta \circ \alpha : G_1 \circ F_1 \Rightarrow G_2 \circ F_2 : C \to E.$$

The diagram below illustrates the assignment we want to define:



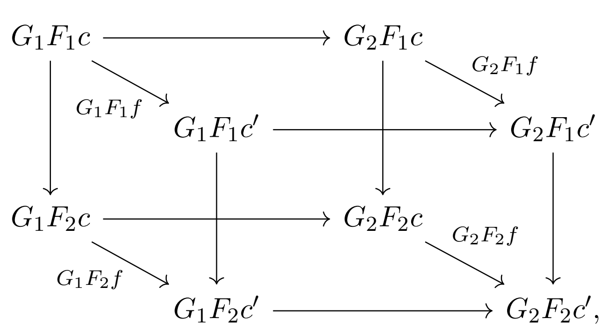
In other words, for each $c \in \mathrm{ob}\, C$, we need a morphism $(\beta \circ \alpha)c : G_1F_1c \to G_2F_2c$ in E. Applying α to c, we get a morphism $\alpha c : F_1c \to F_2c$ in D. There are two ways to send this morphism to one in E, namely, via G_1 and G_2. These two morphisms are connected by β, as in the following diagram (which commutes since β is a natural transformation):

$$\begin{array}{ccc} G_1F_1c & \xrightarrow{\beta(F_1c)} & G_2F_1c \\ {\scriptstyle G_1(\alpha c)}\downarrow & & \downarrow{\scriptstyle G_2(\alpha c)} \\ G_1F_2c & \xrightarrow[\beta(F_2c)]{} & G_2F_2c, \end{array} \tag{1}$$

This diagram gives us a unique diagonal map $(\beta \circ \alpha)c : G_1F_1c \to G_2F_2c$ defined as either of the two compositions:

$$(\beta \circ \alpha)c := (\beta F_2c)(G_1\alpha c) = (G_2\alpha c)(\beta F_1c).$$

In order to show that $\circ$ is well-defined, we need to show that $\beta \circ \alpha$ is a natural transformation $G_1F_1 \Rightarrow G_2F_2$. Let $f : c \to c'$ be a morphism in C. Then we get a cube of morphisms in E:



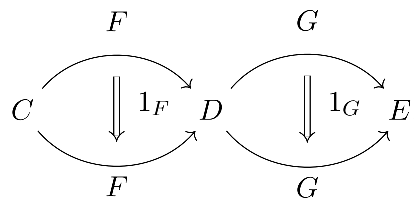
where the back and front faces are the commutative squares defining $(\beta \circ \alpha)c$ and $(\beta \circ \alpha)c'$ as in (1), and the four maps connecting the corners of the two squares are the four compositions of functors applied to the map f. The left and right squares are G_1 and G_2 applied to a square in D which commutes by naturality of α. The top and bottom squares commute by naturality of β. The naturality property of $\beta \circ \alpha$ is commutativity of the (diagonal) square formed by the maps $(\beta\alpha)c$, $(\beta\alpha)c'$, $G_1 F_1 f$, and $G_2 F_2 f$. Commutativity of this square follows from commutativity of all six faces of the cube.

Thus, the assignment defining the horizontal composition functor $\circ$ is well-defined; we need to show that it is a functor.

First, let us check that $\circ$ preserves identity 2-cells, i.e.

$$1_G \circ 1_F = 1_{G \circ F},$$

where 1_F and 1_G denote the identity natural transformations as in the following diagram:



We can apply the defining diagram as in (1):

$$\begin{array}{ccc} GFc & \xrightarrow{1_G(Fc)} & GFc \\ {\scriptstyle G(1_F c)}\downarrow & & \downarrow{\scriptstyle G(1_F c)} \\ GFc & \xrightarrow[1_G(Fc)]{} & GFc. \end{array}$$

Note that every edge in the diagram is the identity, so the diagonal is as well.

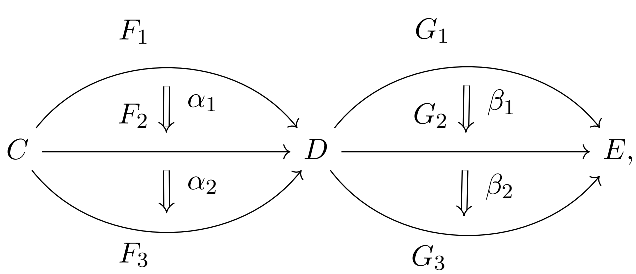
Next, we need to show that horizontal composition preserves vertical composition. We'll use the notation $\cdot$ for vertical composition (i.e. composition of natural transformations within the functor categories) in order to distinguish it from horizontal composition, $\circ$. With this notation, the functoriality property is

$$\circ((\beta_2, \alpha_2) \cdot (\beta_1, \alpha_1)) = (\beta_2 \circ \alpha_2) \cdot (\beta_1 \circ \alpha_1),$$

where $\circ((\beta_2, \alpha_2) \cdot (\beta_1, \alpha_1)) = \circ(\beta_2 \cdot \beta_1, \alpha_2 \cdot \alpha_1) = (\beta_2 \cdot \beta_1) \circ (\alpha_2 \cdot \alpha_1)$. So functoriality amounts to showing

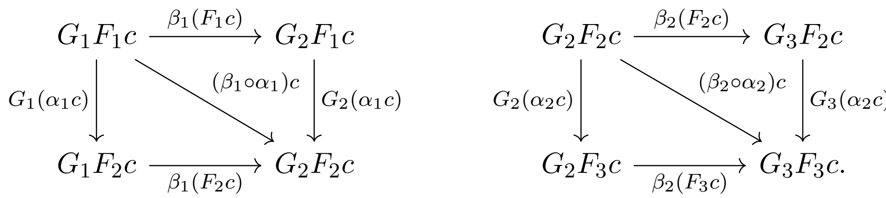
$$(\beta_2 \cdot \beta_1) \circ (\alpha_2 \cdot \alpha_1) = (\beta_2 \circ \alpha_2) \cdot (\beta_1 \circ \alpha_1). \tag{2}$$

We can depict this as a diagram:

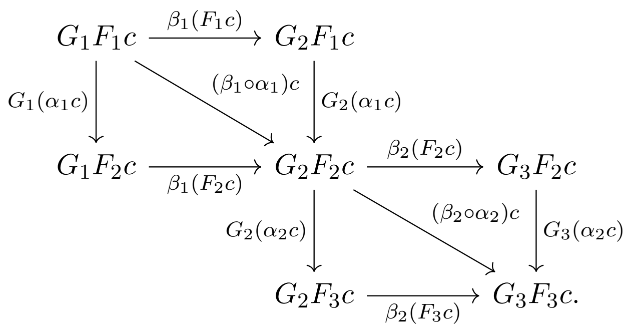


where the words "horizontal" and "vertical" correspond to the respective directions in the diagram, and the parentheses in equation (2) indicate the order in which to carry out the compositions.

To prove (2), consider the commutative squares which define $\beta_1 \circ \alpha_1$ and $\beta_2 \circ \alpha_2$ (as in (1)):

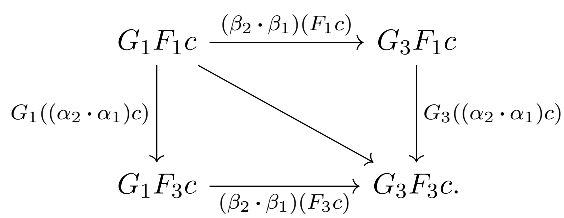


Notice that we can connect these two squares at G_2F_2c to form a single diagram whose diagonal composite is the right side of (2) evaluated at $c \in C$:

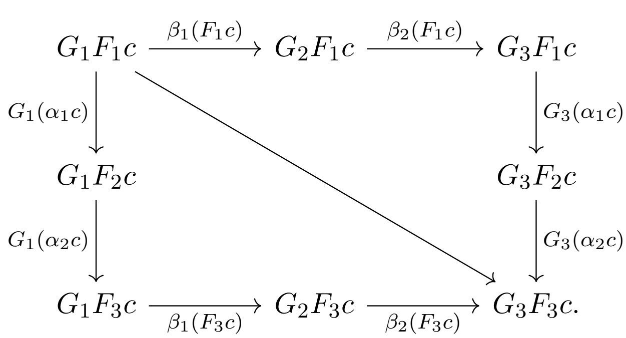


(3)

We can also depict the left side of (2) evaluated at c as the diagonal in the following commutative diagram as in (1):



Since (vertical) composition of natural transformations is defined component-wise (and by invoking functoriality), we can refine the above diagram to:



Notice that we can superimpose the above diagram on the diagram in (3) to get a diagram (which we are not assuming is commutative):

$$\begin{array}{ccccc}
G_1F_1c & \xrightarrow{\beta_1(F_1c)} & G_2F_1c & \xrightarrow{\beta_2(F_1c)} & G_3F_1c \\
{\scriptstyle G_1(\alpha_1 c)}\downarrow & & {\scriptstyle G_2(\alpha_1 c)}\downarrow & ? & \downarrow{\scriptstyle G_3(\alpha_1 c)} \\
G_1F_2c & \xrightarrow[\beta_1(F_2c)]{} & G_2F_2c & \xrightarrow{\beta_2(F_2c)} & G_3F_2c \\
{\scriptstyle G_1(\alpha_2 c)}\downarrow & ? & \downarrow{\scriptstyle G_2(\alpha_2 c)} & & \downarrow{\scriptstyle G_3(\alpha_2 c)} \\
G_1F_3c & \xrightarrow[\beta_1(F_3c)]{} & G_2F_3c & \xrightarrow[\beta_2(F_3c)]{} & G_3F_3c,
\end{array}$$

where the small squares marked with a ? are not yet known to commute. Notice that these squares are also instances of horizontal composition. Namely, the top right square is $(\beta_2 \circ \alpha_1)c$, and the bottom left square is $(\beta_1 \circ \alpha_2)c$. Thus, since all the small squares commute, the entire diagram commutes. So there is only one diagonal composite $G_1F_1c \to G_3F_3c$. This proves that horizontal composition is a functor.

Now we have to show that the data 1-4 satisfy the conditions of a strict 2-category.

I. The identity 1-cells are identity functors, which we know are strict left and right units with respect to composition of functors.

   Given functors $F, G : C \to D$, and a natural transformation $\alpha : F \Rightarrow G$, we need to show that

$$\alpha \circ 1_{1_C} = \alpha \quad \text{and} \quad 1_{1_D} \circ \alpha = \alpha.$$

   In each case, we apply the defining diagram (1):

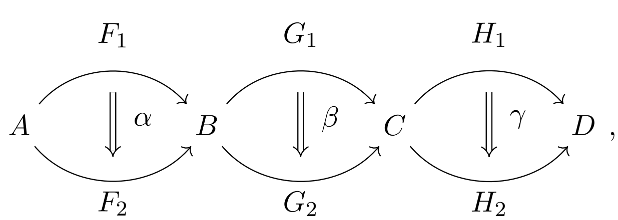
$$\begin{array}{ccc}
F1_Cc & \xrightarrow{\alpha(1_Cc)} & G1_Cc \\
{\scriptstyle F(1_{1_C}c)}\downarrow & & \downarrow{\scriptstyle G(1_{1_C}c)} \\
F1_Cc & \xrightarrow[\alpha(1_Cc)]{} & G1_Cc,
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
1_DFc & \xrightarrow{1_{1_D}(Fc)} & 1_DFc \\
{\scriptstyle 1_D(\alpha c)}\downarrow & & \downarrow{\scriptstyle 1_D(\alpha c)} \\
1_DGc & \xrightarrow[1_{1_D}(Gc)]{} & 1_DGc,
\end{array}$$

   which simplify to

$$\begin{array}{ccc}
Fc & \xrightarrow{\alpha c} & Gc \\
{\scriptstyle 1_{Fc}}\downarrow & & \downarrow{\scriptstyle 1_{Gc}} \\
Fc & \xrightarrow[\alpha c]{} & Gc,
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
Fc & \xrightarrow{1_{Fc}} & Fc \\
{\scriptstyle \alpha c}\downarrow & & \downarrow{\scriptstyle \alpha c} \\
Gc & \xrightarrow[1_{Gc}]{} & Gc,
\end{array}$$

   so both of the diagonal maps are αc.

II. We know that composition of functors is strictly associative. We need to check that horizontal composition of natural transformations is also stricly associative. Given



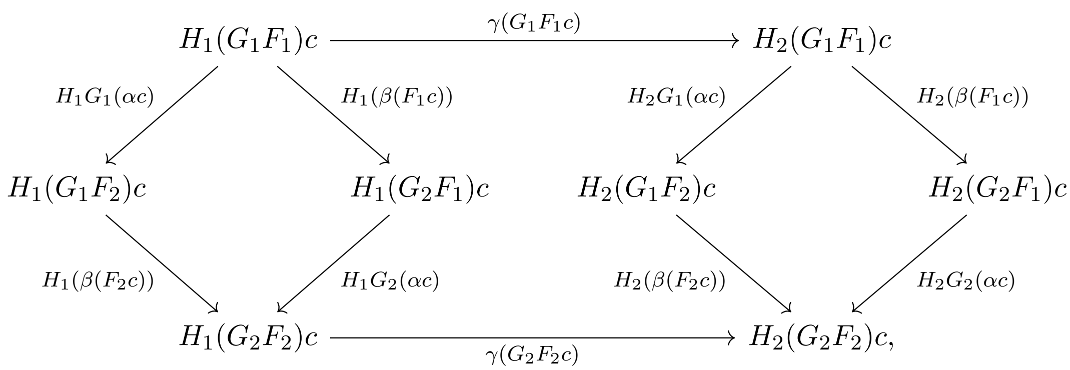
   and $c \in C$ we can form the two diagrams which define $(\gamma \circ (\beta \circ \alpha))c$ and $((\gamma \circ \beta) \circ \alpha)c$:

$$(\gamma \circ (\beta \circ \alpha))c : \quad \begin{array}{ccc} H_1(G_1F_1)c & \xrightarrow{\gamma(G_1F_1c)} & H_2(G_1F_1)c \\ {\scriptstyle H_1((\beta\circ\alpha)c)}\downarrow & & \downarrow{\scriptstyle H_2((\beta\circ\alpha)c)} \\ H_1(G_2F_2)c & \xrightarrow[\gamma(G_2F_2c)]{} & H_2(G_2F_2)c, \end{array}$$

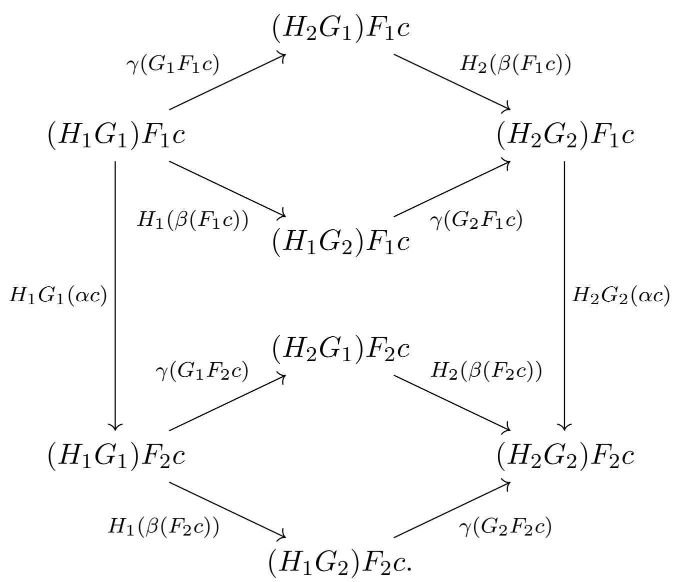
$$((\gamma \circ \beta) \circ \alpha)c : \quad \begin{array}{ccc} (H_1G_1)F_1c & \xrightarrow{(\gamma\circ\beta)(F_1c)} & (H_2G_2)F_1c \\ {\scriptstyle H_1G_1(\alpha c)}\downarrow & & \downarrow{\scriptstyle H_2G_2(\alpha c)} \\ (H_1G_1)F_2c & \xrightarrow[(\gamma\circ\beta)(F_2c)]{} & (H_2G_2)F_2c. \end{array}$$

We can now expand all instances of horizontal compositions of natural transformations into their defining commutative squares, i.e.

$(\gamma \circ (\beta \circ \alpha))c :$



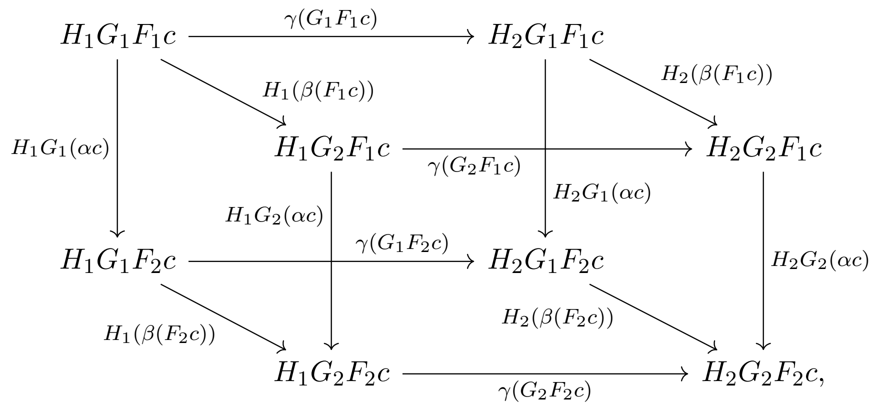
$((\gamma \circ \beta) \circ \alpha)c :$



We can now make the following observations:

- both of the above diagrams have the same eight vertices;
- each diagram can be interpreted as a cube with two of its edges missing;
- whenever two vertices are connected by a map in both diagrams, these maps coincide;
- each diagram contains the two missing edges of the other;

- superimposing the two diagrams on top of each other, aligning the common vertices and edges, yields a commutative cube:



We conclude that both diagonals are the diagonal of the same commutative cube, so they are equal. This proves that horizontal composition is associative.

Thus, categories, functors, and natural transformations define a strict 2-category.